Solving of the Hele-Show's Problem in Presence of the Gap between Impermeable Surfaces

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Abstract[1]

The problem of definition of the function potential, which satisfy to the Laplace's equation in some area on which borders the potential has piecewise-constant values is under consideration. One of boundaries of area is moving. Velocity of movement of boundary is proportional to a gradient of potential. This problem has physical interpretation as at moving of a fluid in porous medium, and at electrochemical shaping.

1. Introduction

The plane problem is under consideration. It has the practical application at designing technological processes of manufacturing of stamps, press, etc. The given problem was solved earlier [1]. However qualitative results, from the point of view of the theory of electrochemical processing, could not be obtained because of crude of a numerical method. The offered numerically-analytical method for solve of a problem allows to carry out research of long processes of working with high accuracy.

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2. Problem Statement

2.1. The Description of Investigated Process

At fig. 1a the scheme of interelectrode space (IES), filled by electrolyte with electro conductivity κ is represented, where $E'CF'$ is the fixed flat electrochemical machining electrode (EME), EGA and BHF is the insulator, ADB is the processable surface. The distance between EME and an insulator l gets out as the characteristic size. The size of gap GH is equal 2a. ADB - the work surface (WS). In the beginning of process flat WS makes the uniform straight line with the insulator. (The thin layer of isolation is put on the surface of metal.) In the further WS is dissolved under action of the electric current. Which arises at connection to EME and WS a voltage source U or a current source I. (For the flat problem the flat current means, i.e. the current proceeding in a cell of the unit cross-section size). Thus there is the dissolution of metal under isolation. That is the object of research.

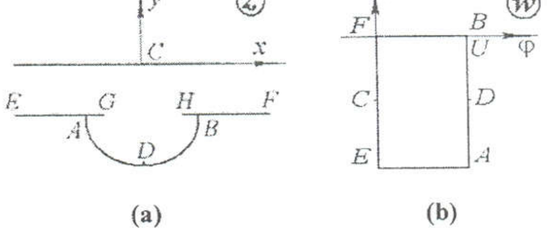
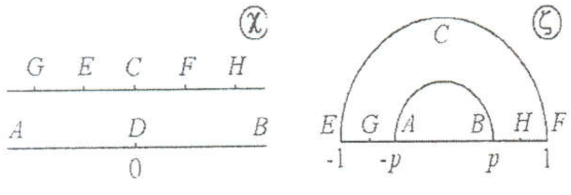


Figure 1. The Form of Area of Interelectrode Space: a) on the Physical Plane; b) on the Plane of Complex Potential

The problem is solving by methods of the theory of functions complex variable. On the plane of complex potential the form of area of interelectrode space looks like a rectangular with width U and height I/κ. Values

U, I can vary in time depending on the characteristic of the used power supply.

The problem is convenient for solving in the parametric form. For this use the plane of parametrical variable $\chi=\sigma+i\upsilon$, the area on which is represented in the form of a belt of unit width (fig. 2a).



(a) (b)

Figure 2. Planes of Parametrical Variables

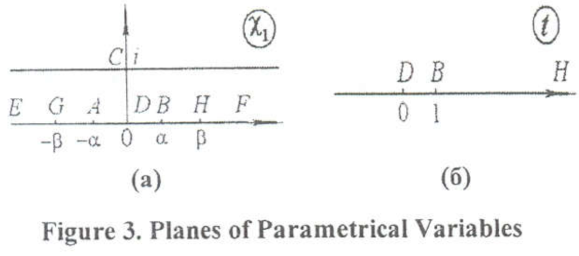
2.2. Basic Equations

The conformal mapping of area of the plane χ on w it is more convenient to spend, using an auxiliary plane ζ. The form of area IES on which looks like a half ring (fig. 2b). Display of an auxiliary plane ζ on a plane of complex potential w it is carried out by the formula

$$w=-\frac{U}{\ln p}\ln\zeta. \quad (1)$$

The auxiliary area in the form of a belt on the plane χ_1 with the points correspondence specified in fig. 3a is entered for an establishment of relation between areas of planes χ and ζ. As segments EA and BF horizontal radiuses of the semiring ζ fit only real values χ_1, function $\chi_1(\zeta)$, according to the principle of symmetry, it is possible to continue analytically on the ring of a plane ζ. Then the mapping ζ on χ_1 we shall search in the form of the sum of the known function having features in points $\zeta=\pm1$ and the convergent Laurent's series in ring:

$$\chi_1=\frac{2}{\pi}\ln\frac{\zeta+1}{1-\zeta}+\sum_{m=1}^{\infty}C_{2m-1}\left(\zeta^{2m-1}+\zeta^{-2m+1}\right). \quad (2)$$



(a) (б)

Figure 3. Planes of Parametrical Variables

At $\zeta=e^{i\xi}$, $0<\xi<\pi$ (2) is become:

$$\chi_1=i+\frac{2}{\pi}\ln\operatorname{tg}\frac{\xi}{2}+2\sum_{m=1}^{\infty}C_{2m-1}\cos(2m-1)\xi, \quad (3)$$

that is top external half-round rings of the plane ζ really is image of the top coast of the belt of the plane χ_1.

At $\zeta=pe^{i\xi}$ (2) is become

$$\chi_1=\frac{4}{\pi}\sum_{m=1}^{\infty}p^{2m-1}\cos(2m-1)\xi+$$
$$+i\frac{4}{\pi}\sum_{m=1}^{\infty}p^{2m-1}\sin(2m-1)\xi+$$
$$+\sum_{m=1}^{\infty}C_{2m-1}\left(p^{2m-1}+p^{-2m+1}\right)\cos(2m-1)\xi+$$
$$+i\sum_{m=1}^{\infty}C_{2m}\left(p^{2m-1}+p^{-2m+1}\right)\sin(2m-1)\xi \quad (4)$$

As image of an internal circle of the ring of the plane ζ is the segment of the real axis on the plane χ_1, then the next equation must be true

$$\frac{4}{\pi}\sum_{m=1}^{\infty}p^{2m-1}\sin(2m-1)\xi+$$
$$+\sum_{m=1}^{\infty}C_{2m-1}\left(p^{2m-1}+p^{-2m+1}\right)\sin(2m-1)\xi=0. \quad (5)$$

From here follows, that

$$C_{2m-1}=-\frac{4}{\pi}\frac{p^{2m-1}}{p^{2m-1}-p^{-2m+1}}.$$

Then

$$\chi_1=\frac{2}{\pi}\ln\frac{\zeta+1}{1-\zeta}+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{p^{4m-2}}{1-p^{4m-2}}\left(\zeta^{2m-1}+\zeta^{-2m+1}\right) \quad (6)$$

In points A and B

$$\chi_1(\pm p)=\pm\alpha=\pm\frac{4}{\pi}\left(\frac{1}{2}\ln\frac{1+p}{1-p}+\right.$$
$$\left.+\sum_{m=1}^{\infty}p^{2m-1}\frac{p^{4m-2}}{1-p^{4m-2}}\left(p^{2m-1}+p^{-2m+1}\right)\right). \quad (7)$$

The conformal mapping of χ on χ_1

$$\chi(\chi_1)=\frac{1}{\pi}\ln\frac{e^{\pi\chi_1}-e^{\pi\alpha}}{1-e^{\pi\alpha}e^{\pi\chi_1}},$$

$$\chi_1(\chi)=\frac{1}{\pi}\ln\frac{e^{\pi\chi}+e^{\pi\alpha}}{e^{\pi\alpha}e^{\pi\chi}+1}. \quad (8)$$

Turning dependence (6), we are receive relation $\zeta(\chi_1)$. Then relation χ and w can be to expressed by the formula:

$$w = -\frac{U}{\ln p} \ln \zeta[\chi_1(\chi)]. \qquad (9)$$

According to (6) and (8) derivative

$$\frac{dw}{d\chi} = \frac{-U}{\ln p}\frac{1}{\zeta[\chi_1(\chi)]}\frac{d\zeta}{d\chi_1}\frac{d\chi_1}{d\chi} = \frac{-U}{\ln p}\frac{1}{\zeta[\chi_1(\chi)]}\frac{\frac{d\chi_1}{d\chi}}{\frac{d\chi_1}{d\zeta}} =$$

$$= \frac{-U}{\ln p}\frac{\frac{e^{\pi\chi}\left(1-e^{2\pi\alpha}\right)}{\left(e^{\pi\alpha}e^{\pi\chi}+1\right)\left(e^{\pi\chi}+e^{\pi\alpha}\right)}}{\frac{4}{\pi}\left[\frac{\zeta}{1-\zeta^2}+\sum_{m=1}^{\infty}\frac{p^{4m-2}}{1-p^{4m-2}}(2m-1)\left(\zeta^{-2m-1}-\zeta^{-2m+1}\right)\right]} \qquad (12)$$

2.3. Boundary Conditions

The dependence $z(\chi, \tau)$ we shall search in the form of

$$z(\chi, \tau) = \chi_1(\chi, \tau) + z_\Delta(\chi, \tau), \qquad (10)$$

(τ – dimensionless time) so that function $z_\Delta(\chi, \tau) = x_\Delta(\chi, \tau) + iy_\Delta(\chi, \tau)$ had no features near to points E and F, and $\operatorname{Im} z_\Delta(\chi_{E,F}, \tau) = 0$.

To define values of parameters α and β, it is necessary to make system of the equations. As near to points G and H $z(\chi, \tau) = O\left((\chi - \chi_G)^2\right)$, $z(\chi, \tau) = O\left((\chi - \chi_H)^2\right)$, at χ, corresponding $\chi_1 = \pm\beta$ value σ is defined from the following equation:

$$\frac{\partial x}{\partial \sigma} = \frac{\partial \chi_1}{\partial \sigma} + \frac{\partial}{\partial \sigma} x_\Delta(\sigma + i, \tau) = 0. \qquad (11)$$

The second equation we shall obtain, satisfying conditions of equality of value $z(\chi(\beta), \tau)$ to the given value a:

$$z(\chi(\beta), \tau) = \beta + z_\Delta\left(i + \frac{1}{\pi}\ln\frac{e^{\pi\beta} - e^{\pi\alpha}}{e^{\pi\alpha}e^{\pi\beta} - 1}, \tau\right) = a. (12)$$

As a result we obtain system of two equations for definition α and β:

$$\begin{cases} M(\alpha, \beta) + \frac{\partial x_\Delta}{\partial \sigma}(i + \Lambda(\alpha, \beta), \tau) = 0, \\ \beta + x_\Delta(i + \Lambda(\alpha, \beta), \tau) = a, \end{cases} \qquad (13)$$

where

$$M(\alpha, \beta) = \frac{\left(e^{\pi\beta} - e^{\pi\alpha}\right)\left(e^{\pi\alpha} - e^{-\pi\beta}\right)}{e^{2\pi\alpha} - 1},$$

$$\Lambda(\alpha, \beta) = \frac{1}{\pi}\ln\frac{e^{\pi\beta} - e^{\pi\alpha}}{e^{\pi\alpha}e^{\pi\beta} - 1}.$$

This system of the equations is solved on each time step. For substitution in the boundary condition it is necessary to define derivatives $\frac{\partial \alpha}{\partial \tau}$ and $\frac{\partial \beta}{\partial \tau}$. For this purpose we shall differentiate the equations of system (13):

$$\begin{cases} \frac{\partial M}{\partial \alpha}\frac{\partial \alpha}{\partial \tau} + \frac{\partial M}{\partial \beta}\frac{\partial \beta}{\partial \tau} + \frac{\partial^2 z_\Delta}{\partial \sigma^2}\left(\frac{\partial \Lambda}{\partial \alpha}\frac{\partial \alpha}{\partial \tau} + \frac{\partial \Lambda}{\partial \beta}\frac{\partial \beta}{\partial \tau}\right) + \\ \qquad + \frac{\partial^2 z_\Delta}{\partial \tau \partial \sigma} = 0 \\ \frac{\partial \beta}{\partial \tau} + \frac{\partial z_\Delta}{\partial \sigma}\left(\frac{\partial \Lambda}{\partial \alpha}\frac{\partial \alpha}{\partial \tau} + \frac{\partial \Lambda}{\partial \beta}\frac{\partial \beta}{\partial \tau}\right) + \frac{\partial z_\Delta}{\partial \tau} = 0 \end{cases} \qquad (14)$$

From here we shall find required derivatives at $\chi = i + \Lambda(\alpha, \beta)$:

$$\begin{cases} \frac{\partial \alpha}{\partial \tau} = \frac{\frac{\partial z_\Delta}{\partial \tau}\left(\frac{\partial M}{\partial \beta} + \frac{\partial^2 z_\Delta}{\partial \sigma^2}\frac{\partial \Lambda}{\partial \beta}\right) - \frac{\partial^2 z_\Delta}{\partial \sigma \partial \tau}\left(1 + \frac{\partial z_\Delta}{\partial \sigma}\frac{\partial \Lambda}{\partial \beta}\right)}{\left(1 + \frac{\partial z_\Delta}{\partial \sigma}\frac{\partial \Lambda}{\partial \beta}\right)\left(\frac{\partial M}{\partial \alpha} + \frac{\partial^2 z_\Delta}{\partial \sigma^2}\frac{\partial \Lambda}{\partial \alpha}\right) - \frac{\partial z_\Delta}{\partial \sigma}\frac{\partial \Lambda}{\partial \alpha}\left(\frac{\partial M}{\partial \beta} + \frac{\partial^2 z_\Delta}{\partial \sigma^2}\frac{\partial \Lambda}{\partial \beta}\right)} \\ \frac{\partial \beta}{\partial \tau} = -\frac{\frac{\partial z_\Delta}{\partial \tau} + \frac{\partial z_\Delta}{\partial \sigma}\frac{\partial \Lambda}{\partial \alpha}\frac{\partial \alpha}{\partial \tau}}{1 + \frac{\partial z_\Delta}{\partial \sigma}\frac{\partial \Lambda}{\partial \beta}} \end{cases} \qquad (15)$$

The boundary condition on the work surface which follows from the Faraday's law, will reorganize in the form:

$$\frac{\partial}{\partial \tau}\operatorname{Re}\left(\chi_1 + z_\Delta(\sigma, \tau)\right)\frac{\partial y_\Delta}{\partial \sigma} - \frac{\partial y_\Delta}{\partial \tau}\frac{\partial}{\partial \sigma}\operatorname{Re}\left(\chi_1 + z_\Delta(\sigma, \tau)\right) = \operatorname{Im}\frac{dw}{d\chi} \qquad (16)$$

where the right part is defined according to the formula (12),

$$\frac{\partial}{\partial \tau}\operatorname{Re}(\chi_1) = \frac{e^{\pi\sigma}\left(1 - e^{2\pi\alpha}\right)}{\left(e^{\pi\sigma} + e^{\pi\alpha}\right)\left(e^{\pi\sigma}e^{\pi\alpha} + 1\right)}\frac{\partial \alpha}{\partial \tau}.$$

3. Numerical Method

3.1. Representation of Required Function

The function which mapping the plane χ on the physical plane is searches in the form of the sum (10).

The function $z_\Delta(\chi)$ it obtains as follows. We search for the solution on border χ=σ in mesh points σ_m $(m=0,\dots,n)$. Values $\operatorname{Im} z_\Delta(\sigma_m) = y_m$ will be required. At σ=σn we shall accept $\operatorname{Im} z_\Delta(\sigma_n) = 0$, as $z_\Delta(\sigma)$ it is fast (as exhibitor) decreases at σ→∞. Values $\operatorname{Im} z_\Delta(\sigma)$ in intermediate between mesh points we shall find, it is similar [2], with the cubic spline $P(\sigma)$, which have two continuous derivatives.

For restoration of function $z_\Delta(\chi)$ we use Schwarz's formula [2] in view of that $z_\Delta(\chi)$ the analytical function having only real values on straight line Imχ=1. As, by act of symmetry of area IES relative to axis y, $\operatorname{Im} z_\Delta(\sigma)$ even function on σ,

$$z_\Delta(\chi) = \operatorname{sh}(\pi\chi)\int_0^\infty P(\sigma)\frac{d\sigma}{\operatorname{ch}(\pi\sigma)-\operatorname{ch}(\pi\chi)}. \qquad (17)$$

The derivative $dz_\Delta/d\chi$ is defined by differentiation (17):

$$\frac{dz_\Delta}{d\chi}(\chi) = -i2\operatorname{ch}\pi\chi\int_0^\infty \frac{dP}{d\sigma}(\sigma)\frac{\operatorname{sh}\pi\sigma}{\operatorname{ch}^2\pi\sigma-\operatorname{ch}^2\pi\chi}d\sigma \qquad .(18)$$

With used the formulas similar (17), (18) and spline $S(\sigma)$ which constructed on values $\operatorname{Im}\frac{\partial z_\Delta}{\partial\tau}(\sigma_m) = f_m$ in mesh points, functions $\frac{\partial z_\Delta}{\partial\tau}(\sigma_m)$ and $\frac{\partial^2 z_\Delta}{\partial\tau\partial\sigma}(\sigma_m)$ used in a boundary condition (16) are defined.

3.2. Algorithm of the Solve of a Non-Stationary Problem

Initial values $z_\Delta(\sigma)$, generally speaking, are equal to zero. Thus from (12) follows, that α=β=a. However, thus arises singularity in points A and B (conterminous with G and H), relate with infinite value of intensity in these points. According to the researches [3] lead earlier, it is known, that in such cases non-stationary process develops self-similarly (that is, geometrical similarity of vicinity IES adjoining to a special point) is kept. In this case initial values are select with the formula (19) which follows. The scale of the deviation of this form from the straight line gets out small enough in comparison with distance up to EME.

Further under formulas (17) and (18) values are defined

$$z_\Delta(\sigma_m) \text{ и } \frac{\partial z_\Delta}{\partial\sigma}(\sigma_m).$$

Values α, β, $\frac{\partial\alpha}{\partial\tau}$ and $\frac{\partial\beta}{\partial\tau}$ also are from (13) and (15).

At substitution of these values in a boundary condition (16) the system of the linear algebraic equations concerning parameters f_m is formed.

Thus, we come to the method of collocations for definition of partial derivatives $\frac{\partial z_\Delta}{\partial\tau}$ and $\frac{\partial^2 z_\Delta}{\partial\tau\partial\sigma}$. Then the step on time is made by Euler's method of the second order of accuracy therefore new values y_m turn out. Further process repeats.

4. Numerical Results

In [1] it has been noticed, that formation of a surface practically does not depend from the given initial form of the work surface. It has not been explained in any way and only subsequently has been found out, that it is related with existence of the self-similar solution 1, which is point of attraction (that is the solutions having the various initial form) aspire to it. It takes place even in the event that process cannot develop infinitely, as in the given problem (at a final gap between EME and isolation).

The given self-similar solution has been obtained in the assumption of significant moving away EME from isolation in comparison with distance $AG=HB$. This solution is defined from the formula

$$z(t) = \frac{1}{3}\frac{8t^2-1-4t-8t^{1.5}\sqrt{t-1}}{t}, \qquad (19)$$

where t - the parameter changing in top half-plane (fig. 3b). The form of this surface is represented on fig. 4.

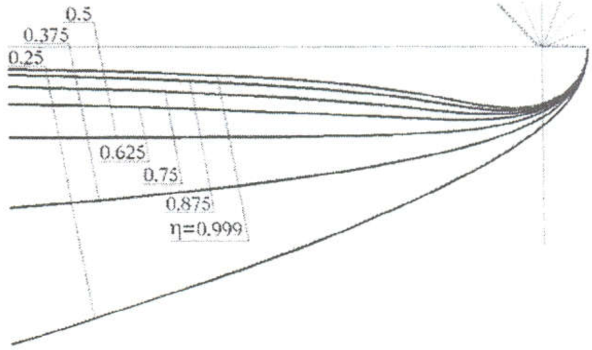


Figure 4. Form of the Self-Similar Surface 1

Any self-similar process is characterized by the constant $\lambda = \frac{l^2\alpha}{k\eta U} = \frac{l^2\alpha\kappa}{k\eta I}$, where $\alpha = \frac{1}{l}\frac{dl}{d\tau}$ - relative speed of change of the characteristic size l (for this solution l is HB), k, η - electrochemical constant and the output on the current. For the given problem λ=0,375.

As a result of numerical researches it is established, that the further development of process occurs under the following scheme. In process of increase in distance HB in comparison with size of the gap in isolation GH the form of the work surface departs from the initial self-similar form, but comes nearer to other self-similar solution 2. This solution has been obtained in the assumption, that surfaces EME and isolation merge in one straight line. This solution is expressed by the formula

$$z(t)=\frac{1}{2}\left(2-i\sqrt{t-1}\ln\frac{t}{\left(\sqrt{t-1}+i\right)^2}\right). \qquad (20)$$

The form of this surface is represented on fig. 5, value $\lambda=4/\pi^2$.

When distance *HB* becomes enough greater in comparison with gap *GH*, the third self-similar form which corresponds dot EME is formed. This form represents half-round with the center located in EME.

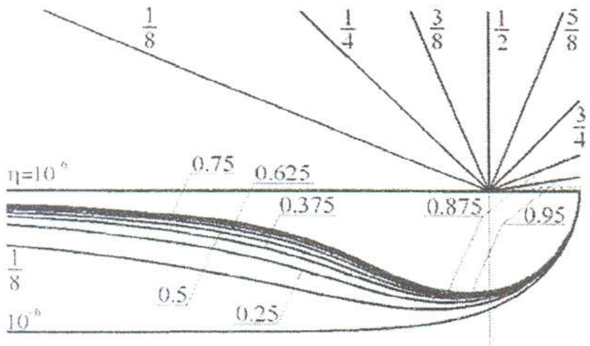


Figure 5. Form of the Self-Similar Surface 2

At the solve of the problem with the insulator of finite thickness this system of self-similar solutions is supplemented with one more (fig.5) which arises at the beginning of process at distances *HB* which small in comparison with thickness of isolation

$$z(\zeta)=\frac{1}{2}\left(2-i\sqrt{\zeta-1}\ln\frac{\zeta}{\left(\sqrt{\zeta-1}+i\right)^2}\right). \qquad (21)$$

In the further the stages considered above repeat all.

5. Conclusions

The effective numerical method allowing with accuracy of the order 10^{-3}-10^{-5} to define form of a work surface practically indefinitely (up to an establishment of the final form of a surface) is developed.

By means of this method there was possible an establishment of the basic laws of shaping a work surface at the given kind of processing. It has appeared, that process is three or four-stage, that is there is a transition from one self-similar form to another.

References

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