Invariant Solution of the no Stationary Elastic Deformation of Hollow Cylinder Problem

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Abstract[1]

Consider the elementary case of the no stationary elastic deformation of hollow cylinder problem under action of internal and external pressure. In exact statement is gotten the problem with free boundaries. Using group of the transformations admitted by the equation some decisions of a problem dependent on infinite number of constant are found.

1. Introduction

Consider the equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + r^{-1} v\right) \qquad (1)$$

with boundary conditions

$$v(t, R_i(t)) = R_i(t) - R_i(t_0); (i = 1, 2),$$

$$p_i(t) + (\lambda + 2\mu)\frac{\partial v}{\partial r} + \lambda R_i^{-1} v = 0,$$

where $r = R_i(t); (i = 1, 2)$ and initial data

$$v\big|_{t=t_0} = v_0(r);\ v_t\big|_{t=t_0} = v_1(r);\ R_i\big|_{t=t_0} = R_i(t_0).$$

We use given values

$$p_i(t),\ p_i(t_0) = p_{i0},\ R_i(t_0) = R_{i0},\ \lambda, \mu, p_0$$

and coordinated conditions $v_0(R_{i0}) = 0,\ (i = 1, 2)$.

According to the above group classification, the equation (1) admits the operators:

$$\begin{aligned} &X_1 = \partial_t; \\ &X_2 = t\partial_t + r\partial_r - \tfrac{1}{2} v\partial_v; \\ &X_3 = \tfrac{1}{2}(t^2 + r^2)\partial_t + r\partial_r - \tfrac{1}{2} tv\partial_v; \\ &X_4 = v\partial_v; \\ &Y_g = g(t,r)\partial_v;\ g_{tt} = g_{rr} + r^{-1} g_r - r^{-2} g \end{aligned} \qquad (2)$$

2. Problem Setting

Here we consider invariants solutions with respect to three different two-dimensional sub algebras of the algebra (2):

$$A.\ X_2 + \alpha X_4;$$
$$B.\ X_3 + \alpha X_4$$
$$C.\ X_1 + X_3 + \alpha X_4;\ .$$

A. $X_2 + \alpha X_4$ Invariant Solutions Class

Invariant solution has the form

$$v = rt^{\nu+1} y(x),\ x = r^2 t^{-2},\ \nu = \alpha - 5/2$$

with special boundary

$$p_j(t) = -2(\mu + \lambda) t^{\alpha - 3/2} y\left(R_j^2 t^{-2}\right) - -(\lambda + 2\mu) R_j^2 t^{\alpha - 7/2} y'\left(R_j^2 t^{-2}\right),\ j = 1, 2,$$

where $R_j t^{\alpha - \frac{1}{2}} y\left(R_j^2 t^{-2}\right) = R_j - R_{j0}$

and special initial data

$$v_0(r) = r t_0^{\alpha - 3/2} y\left(r^2 t_0^{-2}\right),$$

$$v_1(r) = (\alpha - 3/2) r t_0^{\alpha - 5/2} y\left(r^2 t_0^{-2}\right) - 2 r^3 t_0^{\alpha - 9/2} y'\left(r^2 t_0^{-2}\right).$$

We obtain the problem on characteristic value by the equation

$$x(1-x)y'' + \left(\left(\nu - \tfrac{1}{2}\right)x + 2\right)y' - \tfrac{1}{4}\nu(\nu+1)y = 0.$$

Here $y(a) = y(b) = 0$, so that $a = R_{10}^2 t_0^{-2},\ b = R_{20}^2 t_0^{-2}$.

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B. $X_3 + \alpha X_4$ Invariant solutions class

Invariant solution has the form

$$v = r^{-1/2} e^{\frac{-2a}{t+r}} \varphi(q),$$

where $q = r - t^2 r^{-1}$, and is determined by the equation

$$q^2 \varphi'' + 2(q + 2\alpha)\varphi' - \tfrac{3}{4}\varphi = 0, \quad \varphi(a) = \varphi(b) = 0,$$

so that $a = R_{10} - \dfrac{t_0^2}{R_{10}^0}$, $b = R_{20} - \dfrac{t_0^2}{R_{20}^0}$

Using the change $\varphi(q) = \psi(p) p^b e^{-2\alpha p}$ with $p = -q^{-1}$, $b(b-1) = \frac{3}{4}$ we obtain the boundary- value problem on characteristic value α^2.

$$p\psi'' + 2b\psi' - 4\alpha^2 p\psi = 0,$$

$$\psi\left(\frac{R_{10}}{R_{10}^2 - t_0^2}\right) = \psi\left(\frac{R_{20}}{R_{20}^2 - t_0^2}\right) = 0$$

The solution's equation is $\psi = pZ_\nu(2\alpha i p)$, where $Z_\nu = C_1 J_\nu + C_2 Y_\nu$ - general solution of Bessel equation, C_1, C_2 – arbitrary constants.

C. $X_1 + X_3 + \alpha X_4$ Invariant Solutions Class

Invariant solution has the form

$$v = r^{-1/2} e^{\sqrt{2}\alpha \operatorname{arctg} \frac{t+r}{\sqrt{2}}} \varphi(p),$$

where $p = \dfrac{t^2 + 2}{r} - r$.

Obtained equation $(p^2 + 8)\varphi'' + 2(p - 2\alpha)\varphi' - \frac{3}{4}\varphi = 0$ with change $\varphi = \eta\left(\frac{1}{2} - \frac{i}{4\sqrt{2}} p\right)$, was led to hyper geometric equation

$$\xi(\xi - 1)\eta'' + \left(2\xi - \left(\tfrac{i}{\sqrt{2}\alpha} - 1\right)\right)\eta' - \frac{3}{4}\eta = 0,$$

with boundary conditions $\eta(a_i) = \eta(b_i) = 0$,

where

$$a_i = \frac{1}{2} - i\frac{1}{4\sqrt{2}}\left(\frac{t_0^2 + R_{10}^2}{R_{10}} - R_{10}\right),$$

$$b_i = \frac{1}{2} - i\frac{1}{4\sqrt{2}}\left(\frac{t_0^2 + R_{20}^2}{R_{20}} - R_{20}\right)$$

References

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