Investment portfolio optimization and some classes of risk measures

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Abstract[1]

The asymmetric financial and actuarial risk measures based on Wang's distortion risk measures, which reflect distinct risk perceptions in the cases of profits and losses, are defined and studied. This difference in terms of risk attitude was established in Kahneman and Tversky's prospect theory. Asymmetric variants of some other risk measures such as Pedersen-Satchell measure, Orlich measure, spectral measure are defined. New risk measures based on Value-at-Risk and Conditional Value-at-Risk wich take into account the weights as left or right tails of profit distribution are defined too. The described measures are applied in portfolio optimization. The results of numerical experiments for these measures are described.

1. Introduction

The existing variety of approaches for financial risk estimation reflects both: the complexity of a market and the diversity of psychological risk perception. In particular, Kahneman and Tversky's prospect theory [1] experimentally shows that the attitudes to the gains and losses being equal in the absolute values have not only «unlike signs». In particular, there are a negligible number of people, wishing to participate in a lottery where equal profit and loss have the same probabilities. The reactions on the financial markets are similar: failure stimulates activity of a company, which can lead to a big success in future. At the same time, success can weaken the successful companies. Applicability of the similar risk measures was mentioned by a number of authors (for example, in [2]).

For an application of asymmetric risk measures such stock return rate should be used that have a set of values symmetric relative to zero. Zero corresponds to a case of stock price stability. The rates of returns such as $\chi_1(n) = \ln\frac{c_{n+1}}{c_n}$ or $\chi_2 = \frac{c_{n+1} - c_n}{c_{n+1} + c_n}$ can be used. Here, c_n is a stock price at day n. The value areas of these rates are $(-\infty,\infty)$ and $[-1, 1)$, respectively. It is important to note, that both rates are dimensionless. These characteristics, for example, wouldn't allow using similar approaches in the actuarial calculations where risk measure is associated with insurance premium, i.e. a dimensional value. Some of the known risk measures' constructions can be adapted to an asymmetric case. Note that often risk measures are defined not for profits, but for losses.

2. The risk measures application to optimal investment portfolio forming

We consider the set of stocks (n kinds), of which the investor forms the portfolio $U = (u_1, \ldots, u_n)$ where u_i is a share of the means spent on the i-th kind of stocks, i.e.

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$u_i \geq 0$, $\sum_{i=1}^{n} u_i = 1$. Let T be the time interval (long enough) for which is observed stock price, τ be a time interval (short), following interval T. Let us choose some family $\Psi_{r,T}(U)$ of measures of the risk, depending on vector parameter $r \in R$ and a portfolio U, $\Psi^*_{r,T}(U)$ is the risk measure which is calculated for historical samples of portfolio profitability for the period T and $\varphi_\tau(U)$ is the profitability of a portfolio for an interval τ . For developing the most effective risk measures the following two optimizing problems must be solved consistently:

$$U_T(r) = \arg\min_U \Psi^*_{r,T}(U), \quad (1)$$

$$r^0(T,\tau) = \arg\max_{r\in R}\left(\varphi_\tau\left(U_T(r)\right)\right). \quad (2)$$

It is expedient to select the set

$$R^*(T,\tau) = \left\{r \in R : \varphi_\tau\left(U_T(r)\right) > \alpha r^0(T,\tau)\right\},$$

where the number α is close to 1 (for example 0,95; 0,99). If this technique gives close sets for the various markets then it is possible to recommend the corresponding risk measures for using it for portfolio forming. If the risk measure $\Psi_{r,T}(U)$ has the minimum value, one can predict high enough income on the subsequent short time interval.

For the using of this technique the investor must:

- to choose time intervals T and τ. (it can be a rupture between them, it is possible to use a technique of a moving average);
- to establish area of change the set R of risk measure parameters;
- to choose an optimization method in the problem (1). If number of stocks n is small (2-10), we applied exhaustion with step 0,1; for the big number of stocks the combination of stochastic optimization with some algorithm of descent was used;
- to choose similarly an optimization method in the problem (2).

3. Complex quantile risk measures

3.1. Definitions

$\text{VaR}_\alpha^-[X]$ is a distribution quantile of X with parameter α. α usually takes values 0,1; 0,05; 0,01 which are the probabilities of the events "X is smaller than $\text{VaR}_\alpha^-[X]$". $\text{CVaR}_\alpha^-[X]$ is a conditional mean of X if X is lower than or equal to $\text{VaR}_\alpha^-[X]$. Different from $\text{VaR}_\alpha^-[X]$, $\text{CVaR}_\alpha^-[X]$ is a coherent risk measure [4]. Both measures take into consideration only the weight of the left tail. For example, let the random value X be uniformly distributed on the set $[-1,1]$ and the random value Y - uniformly distributed on the set $[-1,0]\cup[9,10]$. Then $\text{VaR}_{0,25}^-[X]=\text{VaR}_{0,25}^-[Y]=-0,5$ and $\text{CVaR}_{0,25}^-[X]=\text{CVaR}_{0,25}^-[Y]=-0,75$ but financial results are very different (the idea of this example is taken from [5]). At the same time $\text{VaR}_{0,75}^-[X]=0,5$; $\text{VaR}_{0,75}^-[Y]=9,5$.

Thus it is reasonable to consider the weight of the right tail, i.e. to include values $\text{VaR}_\alpha^+[X]$ – $(1-\alpha)$-quantile of the distribution X and $\text{CVaR}_\alpha^+[X]$ – conditional mean of X if X is more than or equal to $\text{VaR}_\alpha^+[X]$. Therefore it makes sense to use the following constructions as risk measures:

$$\text{VaR}_{\alpha,\beta,k}^{\pm}[X] = k\cdot\text{VaR}_\alpha^-[X] + \text{VaR}_\beta^+[X], \quad (3)$$

$$\text{CVaR}_{\alpha,\beta,k}^{\pm}[X] = k\cdot\text{CVaR}_\alpha^-[X] + \text{CVaR}_\beta^+[X]. \quad (4)$$

Both risk measures (3), (4) depend on three numerical parameters.

3.2. Empirical results

We analyzed a few sets of stocks for different T and τ. It was established that:

- results are better for risk measure (4) then for risk measure (3);
- for $\alpha=\beta=0,05$ the best results have been achieved if $k\in(0,4;0,8)$ (risk measure (3)) and if $k\in(0,8;1,2)$ (risk measure (4)).

4. Asymmetric distortion risk measures

4.1. Definitions

Distortion risk measures were introduced to estimate actuarial risks [6]. These measures are based on a Choquet integral construction. Let us consider the definition of the asymmetric distortion risk measure (ADRM) [7]. Let $\overline{g}(t) = \left(g_1(t), g_2(t)\right)$ be a pair of non-decreasing functions $g_i : [0,1] \to [0,1]$. We define the asymmetric distortion risk measure by the following equation:

$$\Psi_{\overline{g}}(X) = \int_{-\infty}^{0}\left[1 - g_1\left(\overline{F}_X(t)\right)\right]dt - \\ -\int_{0}^{\infty} g_2\left(\overline{F}_X(t)\right)dt, \quad (5)$$

where $\overline{F}_X(t) = 1 - F_X(t)$ is the additional distribution function of risk. This risk measure is applicable to both indicators χ_1 and χ_2 due to properties of the function $\overline{F}_X(t)$. Standard distortion risk measures correspond to the case $g(t) = g_1(t) = g_2(t)$. It is worth to note, that quantile risk measures VaR, CVaR are the special cases of the distortion risk measures when a function $g(t)$ is appropriately chosen.

4.2. Some properties of ADRM

1. ADRM doesn't depend on a risk X itself but only on its underlying distribution.

2. If $\overline{g}(t)=(t,t)$, then $\Psi_{\overline{g}}(X)=-E[X]$ where E is the mean (it is assumed that the mean exists).

3. ADRM is positive for the guaranteed losses and negative for the guaranteed returns ($\overline{F}_X(t)=0$ if $t>0$ $\overline{F}_X(t)=1$ if $t<0$).

4. ADRM is no additive in the general case.

5. There is an important special case when risks *X, Y* are comonotonic which always take on the values of the same signs. Recall that random variables are comonotonic if the increasing of one involves the increasing of the other variable. ADRM is additive for such risks.

6. ADRM is monotonous: if almost surely $X \le Y$, then $\Psi_{\overline{g}}(X) \ge \Psi_{\overline{g}}(Y)$. It follows from the inequality $\overline{F}_X(t) \le \overline{F}_Y(t)$ validity for any *t*.

7. ADRM is positively homogeneous: if $\lambda \ge 0$ then $\Psi_{\overline{g}}(\lambda X)=\lambda \Psi_{\overline{g}}(X)$.

8. ADRM is functionally invariant under translations [8]: the function $\Psi_{\overline{g}}(X+a)$ with a determined variable a is continuous and non-increasing.

9. Assume that $\overline{g}^*(t)=\left(1-g_2(1-t),1-g_1(1-t)\right)$. Then $\Psi_{\overline{g}}(-X)=-\Psi_{\overline{g}^*}(X)$.

10. ADRM is not sub – or superadditive even for convex functions $\left(g_1(t),g_2(t)\right)$. Consequently ADRM is not coherent.

4.3. Empirical results

There were examined the pairs of functions

$$\overline{g}(t)=\left(t^k,t^s\right), \tag{6}$$

$$\overline{g}(t)=\left(1-(1-t)^k,1-(1-t)^s\right). \tag{7}$$

For 7 indexes (ASX200, XAX, BSE_SENTEX, BVSP, CAC40, CSI200, FTSE EUROTOP 100), T=364 days (a year), τ=2 days, exhaustion for portfolios with step 0,17 (the total number equals 924), exhaustion for pairs (k,s)∈(0,3]× (0,3] with step 0,2 (the total number is 225). Optimal (in sense of p. 2) are 51 pairs (k,s) for case (6) and 45 pairs for case (7).

- For pairs (6) among them k<s in 6 cases, k=s in 2 cases, k>s in 43 cases.
- For pairs (7) among them k<s in 25 cases, k=s in 1 case, k>s in 19 cases.

5. Other asymmetric risk measures

5.1. Pedersen-Satchell's asymmetric risk measure

Pedersen and Satchell [9] defined a broad class of risk measures, some particular cases of this class are well known and widely used. It admits asymmetric variant:

$$\Psi(X)=\left[\int_{-p}^{0}\left|t-b^-\right|^{\alpha^-}\cdot W^-\left(F_X(t)\right)dF_X(t)\right]^{\theta^-}+$$

$$+\left[\int_{0}^{p}\left|t-b^+\right|^{\alpha^+}\cdot W^+\left(F_X(t)\right)dF_X(t)\right]^{\theta^+}.$$

The parameter p takes values 1 or ∞ depending on the return rate. In general case, risk measure depends on six numerical and two functional parameters.

5.2. Asymmetric Orlich type risk measures

Orlich type risk measures were introduced only for positive risks [10], in [11] the symmetric variant for alternating-sign case was proposed. Asymmetric variant is the following. Let function Φ: (−∞,∞)→(−∞,∞) (generalized normalized Jung function) have the following properties: Φ(0)=0, Φ(1)=1, Φ(−1)=−1, it is increasing, convex on a positive semiaxis and concave on a negative. Then a risk measure can be defined by the equation $E\left[\Phi\left(\dfrac{X}{\Psi(X)}\right)\right]=\alpha\in(0,1].$. It depends on one numerical and one functional parameter. Function Φ is odd in standard case.

5.3. Asymmetric spectral risk measures

Spectral risk measures were introduced in [12]. Let us define its asymmetric analogue. Let $\varphi^+(u),\varphi^-(u)$ be non-increasing functions such as $\varphi^+(0)=\varphi^-(0)=0$, $\varphi^+(1)=\varphi^-(1)=1$ and $q_u(X)=F_X^{-1}(u)$ (in continuous case) is distribution u-quintile. Let $p^*=P[X<0]$. Then asymmetric spectral risk measure is determined as

$$\Psi(X)=-A\left(\int_0^{p^*}\varphi^-(u)q_u(X)du+\int_{p^*}^{1}\varphi^+(u)q_u(X)du\right),$$

where $A=1/\left(\int_0^{p^*}\varphi^-(u)du+\int_{p^*}^{1}\varphi^+(u)du\right)$ is a normalizing factor.

It is reasonable to apply this measure to χ_2 return rate, which allows avoiding the problems with convergence of integral. This risk measure depends on two functional parameters. The traditional spectral measure complies with the case $\varphi^+(u)=\varphi^-(u)$.

6. Conclusion

A few classes of financial risk measures are defined. In particular some of them take into account the asymmetry influence (in spirit of Twerski-Kanneman prospect theory). The procedure of checking risk measure efficiency for portfolio formation and determination of the best parameters is described. The results of numerical experiments are given for some of defined risk measures. It is shown that the using of defined risk measures leads to good stocks portfolios.

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