New sufficient conditions for the integer round-up property for 1D cutting stock problem

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Abstract[1]

The integer round-up property of the linear cutting stock problem is investigated. An algorithm for verifying the sufficient conditions under which the problem has no integer round-up property is designed.

1. Introduction

Let us consider the classical 1-dimensional Cutting Stock Problem (1CSP): 1-dimensional material objects of a given length L are divided into smaller pieces of desired lengths $l_1, l_2, \ldots, l_n$ in order to fulfill the order demands $b_1, b_2, \ldots, b_n$. The goal is to minimize the total waste. For short we denote 1CSP problem as P=(L,n,l,b).

It is well-known fact that 1CSP can be simulated as a Linear Integer-valued Optimization Problem [1]. Any feasible cutting pattern can be represented by n-dimensional nonnegative vector $a_j=(a_{1j},\ldots, a_{nj})^T$ satisfying $\sum_{i=1}^{n} l_i a_{ij} \le L$. Denote by an integer number x_j the number of stock material objects to be cut according the pattern a_j, and at the same time

$$z=\sum_{j=1}^{N} x_j \to \min, \text{s.t. } \sum_{j=1}^{N} a_{ij} x_j \ge b_i, i=1,\ldots n,$$

$x_j \ge 0$, integer, $j=1,\ldots,N$, where N denotes the number of cutting patterns.

This model can be written in the short form:

$$z=\sum_{j=1}^{N} x_j \to \min, \text{s.t. } Ax \ge b, x \in Z_{+}^{n}, \quad (1)$$

here matrix A contains all maximal cutting patterns as the columns. Let Z*(P) be an optimal value of the 1CSP (1).

The Linear Programming (LP) relaxation provides a lower bound of 1CSP solution:

$$z=\sum_{j=1}^{N} x_j \to \min, \text{s.t. } Ax \ge b, x \in R_{+}^{n}, \quad (2)$$

Let $Z_s(P)$ be an optimal value of the (2). Now we shall give the following definition.

Definition 1. *The problem P is called possessing the IRUP (integer round-up property) if* $Z^*(P)=\lceil Z_s(P) \rceil$.

The numerous operations have shown that the most of 1CSP problems possess the IRUP property [2]. Due to this fact at the present time there are a lot of algorithms that permit to get the optimal solutions for the 1CSP problems with the great numbers of items. At that they use the value $\lceil Z_s(P) \rceil$ as the lower bound for Z*(P) and try to get the reaching it solution (see [3], [4]). However in the case if this solution can't be obtain, the natural question is appear: has the present problem IRUP property or not? The answer of this question may be given either making the Complete Enumeration of every possible feasible cutting patterns or using the Integer Programming algorithms [3]. Both at the first and at the second cases it requires the considerable time consumption.

At the present paper we suggest the effective algorithm that allows us partially overcome above difficulties. For the certain problems during the acceptable time we succeeded in displaying that they don't possess the IRUP property.

2. The main theory

Let us allow $x_t=0$ for the certain fixed $0<t\le N$ and then construct a new problem:

$$z=\sum_{j=1}^{N} x_j \to \min, \text{s.t. } Ax \ge b, x_t=0, \ x \in R_{+}^{n}. \quad (3)$$

Let $Z_s^t(P)$ denotes the optimal value for problem (3).

Proposition 1. *Let $Z_s^t(P)>Z_s(P)$. Then the corresponding cutting pattern $a_t=(a_{1t},\ldots,a_{nt})^T$ must be presented in any optimal solution of problem (2) with $x_t>0$.*

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Proposition 2. *The cutting pattern $a_t=(a_{1t},...,a_{nt})^T$ must be presented in any solution of problem (2) with $x_t>0$ if* $Z_s^t(P) > \sum_{j=1}^{N} x_j$.

Let we have got the problem P=(L,n,l,b) and the certain pattern a_t. Here we can denote a modified problem P^t=(L,n,l,b-a_t), where vector $b-a_t=(b_1-a_{1t}, \ldots, b_n-a_{nt})$.

Lemma 1. *If for certain problem P exists the cutting pattern $a_t=(a_{1t},...,a_{nt})^T$ such that $Z_s^t(P)>\lceil Z_s(P)\rceil$ and problem P possesses the IRUP property, then the any optimal solution of the problem 1CSP (1) contains the pattern a_t, $x_t\geq 1$ and $Z^*(P)=1+Z^*(P^t)$.*

Proof. It follows from the conditions of Lemma 1 that $Z^*(P)=\lceil Z_s(P)\rceil < Z_s^t(P)$.

Any feasible solution of problem (1) is a feasible solution of problem (2), hence, according to the Proposition 1, the optimal solution of 1CSP must contains the cutting pattern a_t, where $x_t>0$. Taking into account that x_t is the integer number, we obtain $x_t\geq$ 1.Some solution of 1CSP for P^t may be constructed from the optimal solution of 1CSP for P if we remove the pattern a_t with $x_t=1$, hence

$$Z^*(P)-1 \geq Z^*(P^t).$$

And inversely, some solution of 1CSP for P is constructed from the optimal solution of 1CSP for P^t if we add the pattern a_t with $x_t=1$, hence

$$Z^*(P) \leq Z^*(P^t)+1.$$

Comparing the last two formulas give us the equality $Z^*(P)=1+Z^*(P^t)$.

This completes the proof.

Theorem. *If for certain problem P exists the cutting pattern $a_t=(a_{1t},...,a_{nt})^T$ such that $\lceil Z_s^t(P)\rceil > \lceil Z_s(P)\rceil$ and $\lceil Z_s(P)\rceil < 1+\lceil Z_s(P^t)\rceil$ then P is not possesses the IRUP property.*

Proof. Let problem P possesses the IRUP property. Then according to Definition 1

$$\lceil Z_s(P)\rceil = Z^*(P). \tag{4}$$

Substituting (4) in the second condition of this Theorem, we obtain

$$Z^*(P) < 1 + \lceil Z_s(P^t)\rceil \tag{5}$$

It is well-known fact that

$$\lceil Z_s(P^t)\rceil \leq Z^*(P^t). \tag{6}$$

It follows from (5) and (6) that:

$$Z^*(P) < 1 + Z^*(P^t). \tag{7}$$

Taking into account our assumption and the first condition of the Theorem we obtain that the conditions of the Preposition 2 are true.

Hence,

$$Z^*(P) = 1 + Z^*(P^t). \tag{8}$$

Relation (7) contradicts to the relation (8). This completes the proof. Problem P is not possesses the IRUP property.

3. Algorithm of the IRUP test

Algorithm of the IRUP test consists of the following steps:

Step 1. Let $(x_1^0, x_2^0,\ldots, x_N^0)$ be an optimal solution of (2) for the problem P=(L,n,l,b). Maximum n components of this solution are not equal to 0.

Without loss of generality we could consider that the optimal solution has the following type: $(x_1^0, x_2^0,\ldots, x_h^0, 0,\ldots,0)$, where $x_i^0\neq 0$, $i=1,\ldots,h$, $h\leq n$.

Step 2. If there exists the number j ($1\leq j\leq h$) such that $Z_s^j(P)>Z_s(P)$ then we construct a new problem $P'=(L,n,l,b-a^j)$;else **Stop.** Answer: we can't classify the problem P.

Step 3. If $\lceil Z_s(P^j)\rceil > \lceil Z_s(P)\rceil - 1$ then **Stop**. Answer: problem P doesn't possess the IRUP property; else change the problem P:=P' and return to **Step 2.**

Remark. *Let $(x_1^0, x_2^0,..., x_n^0, 0,...,0)$, be an optimal solution of the problem P. Let us consider the following problem:*

$$z=\sum_{j=1}^{N} x_j \rightarrow \min, \text{s.t. } Ax \geq b, \quad x_1^0=0,\ldots,x_n^0=0,\ x\in R_+^n, \tag{9}$$

Let $Z`_s(P)$ be the optimal LP value of the problem (9). If $\lceil Z`_s(P)\rceil = \lceil Z_s(P)\rceil$ then it is obviously that the previous IRUP test is non-effective (it will not give us an answer, either this problem possesses IRUP or not).In other words, even in the case if IRUP test is non-effective, it reduces the number of items in the main problem P.

Example

Problem P =(L=1000,n=52, (l,b) = (693, 1), (668, 1), (667, 1), (650, 1),(624, 1), (616, 1), (614, 1), (606, 1), (600, 1), (581, 1), (578,1), (573, 1), (541, 1), (527, 1), (521, 1), (501, 1), (455, 1),(443, 1), (440, 1), (419, 1), (375, 1), (372, 1), (357, 2), (356,1), (354, 1), (352, 1), (342, 2), (303, 1), (301, 1), (288, 1),(264, 1), (246, 1), (241, 1), (226, 1), (223, 1), (183, 1), (179,1), (177, 1), (168, 1), (164, 1), (163, 1), (162, 2), (156, 1),(137, 1), (135, 1), (121, 1), (97, 1), (95, 1), (74, 1), (68, 1),(66, 1), (55, 1).

This is $Z_s(P)$ = 19.

Let's choose t=19, then a_t = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).

This is Z_s^t (P) = 19.0026.

Hence $Z_s^t(P) > \lceil Z_s(P) \rceil$.

The reduced problem P^t=(L=1000, n= 51,(l,,b)=(693, 1), (668, 1), (667,1), (624, 1), (616, 1), (614, 1), (606, 1), (600, 1), (581, 1),(578, 1), (573, 1), (541, 1), (527, 1), (521, 1), (501, 1), (455,1), (443, 1), (440, 1), (419, 1), (375, 1), (372, 1), (357, 2),(356, 1), (354, 1), (352, 1), (342, 1), (303, 1), (301, 1), (288,1), (264, 1), (246, 1), (241, 1), (226, 1), (223, 1), (183, 1),(179, 1), (177, 1), (168, 1), (164, 1), (163, 1), (162, 2), (156,1), (137, 1), (135, 1), (121, 1), (97, 1), (95, 1), (74, 1), (68,1), (66, 1), (55, 1)).

This is Z_s (P^t) =18.001 *and* $\lceil Z_s(P^t) \rceil > \lceil Z_s(P) \rceil 1$ is true. Conclusion: the initial instance P has not IRUP.

4. Conclusion

The presented algorithm of IRUP test should be used as the primary step for the solution of problem 1CSP.On application of this algorithm the more precise definition of the 1CSP optimal solution's lower border may be occurred: either the lower border is equal to $\lceil Z_s(P) \rceil$, or is equal to $\lceil Z_s(P) \rceil$+1. There are a lot of hard problems with unknown answer: are they possess IRUP property or not. The present algorithm was successfully used for the investigation of these hard problems.

References

1. Mukhacheva E.A., Rubinshtein G.Sh. "*Mathematical Programming*". Nauka, Novosibirsk, Russia, 1987.
2. Marcotte O. "*The cutting stock problem and integer routing*". *Math. program*, 1985; 33/1: 82-92.
3. Scheithauer G., Terno J. "Improving the formulation of the cutting stock problem". In: *Presentation given at the International Symposium on Mathematical Programming*. Lausanne, France, 1997.
4. Mukhacheva E.A., Belov G.N., Kartak V.M., Mukhacheva A.S. "Linear one-dimensional cutting-packing problems: numerical experiments with the sequential value correction method (SVC) and a modified branch-and-bound method (MBB)". *Pesquisa Operacional*, 2000; Vol. 20, No. 2, pp. 153-168.