LP-based volume bounds in orthogonal packing

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Abstract[1]

The d *-Dimensional Orthogonal Packing Problem* (OPP-d, $d \geq 2$) asks whether all given boxes can be orthogonally packed into a given container without rotations. The simplest bound for OPP is the *volume bound*: if the total volume of the items exceeds that of the container then the instance is infeasible. *Conservative scales* (CS) are modified item sizes such that if OPP is feasible, it is also feasible with the modified sizes. Thus, the volume bound for the modified instance is valid for the original instance. Up to now, CS have been constructed either (i) completely independently in each dimension using *dual-feasible functions* or (ii) by an exact method in the 2D case. Between (i) and (ii), there are possible heuristic algorithms which construct d conservative scales (in d dimensions) simultaneously. Our first efforts in this direction have shown that a simple LP iteration produces results nearly identical with the exact method in smaller time. 3D results are presented as well.

1. Introduction

Let there be given a set of d-dimensional items (boxes) that need to be packed into a fixed container. The input data describe the container sizes $W_k \in \mathbb{Z}_+$, $k = \overline{1,d}$, and the sizes of the n items $w_i^k \in \mathbb{Z}_+$, $k = \overline{1,d}$ for each item $i \in V = \{1,\ldots,n\}$. Without loss of generality, we assume that each item fits into the container, i.e., $w_i^k \leq W_k$ holds for each box i and dimension k. The guillotine constraint is not considered. The d *-Dimensional Orthogonal Packing Problem* (OPP-d) [9] asks whether all the boxes can be orthogonally packed into the container without rotations.

OPP is strongly NP-complete and polynomially equivalent to the orthogonal *strip-packing problem* [1]. OPP is a subproblem in solution methods for orthogonal *bin-packing* and *knapsack problems*, cf. [9,2].

Many solution methods use *bounds on the solution value*. For example, the simplest bound for OPP is the *volume bound*: if the total volume of the items exceeds that of the container then the instance is infeasible. Bounds are obtained from *relaxations* of the main problem, i.e., somewhat simpler problems. Bounds should be preferably quickly computable. However, some bounds are so strong that it pays off to spend more time for their computation.

Conservative scales (CS) [10] are modified item sizes such that if OPP is feasible, it is also feasible with the modified sizes. Thus, the volume bound for the modified instance is valid for the original instance. Often it is stronger, which is heavily used in algorithms.

Definition 1. *Let $(W, w) \in \mathbb{Z} \times \mathbb{Z}^n$ be an instance of the 1D binary knapsack problem. Vector $\widetilde{w} \in \mathbb{R}^n$ is a conservative scale (CS) for (W, w) if any feasible solution of (W, w) stays feasible for $(W, \widetilde{w})$:*

$$\forall a \in \{0,1\}^n : \quad \sum_i w_i a_i \leq W \Rightarrow \sum_i \widetilde{w}_i a_i \leq W. \quad (1)$$

Proposition 1. *Let $I = (W_1,\ldots,W_d, w^1,\ldots,w^d)$ be an instance of OPP-d. Let $\widetilde{w}^k$ be a CS for (W_k, w^k), $k = \overline{1,d}$. Then any feasible packing for I can be transformed into one for the instance $\widetilde{I} = (W_1,\ldots,W_d, \widetilde{w}^1,\ldots,\widetilde{w}^d)$.*

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Thus, conservative scales define a new OPP instance whose set of packings is a superset of that for the original instance (if it is the same, we obtain a so-called *equivalent instance*, cf. [8]). Our goal is to maximize its volume bound:

$$\begin{array}{ll} \max & \sum_i \prod_k \widetilde{w}_i^k \\ \text{s.t.} & \widetilde{w}^k \text{ is a conservative scale for } (w^k, W_k). \end{array} \quad (2)$$

Up to now, CS have been constructed either (i) completely independently in each dimension using *dual-feasible functions* (cf. [6] and Definition 2 below), or (ii) optimizing problem (2) exactly in the 2D case [4]. Between (i) and (ii), there are possible heuristic algorithms which construct d conservative scales (in d dimensions) simultaneously.

2. LP heuristics for conservative scales

In this section we discuss previous results on conservative scales, introduce the new heuristics and discuss their properties.

2.1. An overview of conservative scales

Interpretation as valid inequalities. According to Definition (1), consider some constant CS $\widetilde{w}$ for $(W, w) \in \mathbb{Z} \times \mathbb{Z}^n$. Now, the inequality

$$a^{\mathrm{T}} \widetilde{w} \le W$$

is a valid inequality for the knapsack polyhedron

$$P(W, w) = conv\{a \in \{0,1\}^n : a^{\mathrm{T}} w \le W\}. \quad (3)$$

This suggests application of CS in polyhedral theory, cf. [6].

The CS polyhedron. Given all extreme points $a^j \in \{0,1\}^n$, $j = \overline{1,\eta}$ of the knapsack polyhedron (3), any CS $\widetilde{w}$ satisfies the constraints

$$a^{j\mathrm{T}} \widetilde{w} \le W, \qquad j = \overline{1,\eta}, \quad (4)$$

i.e., all CS for a certain (W, w) are given by the polyhedron

$$D(P(W, w)) = \{\widetilde{w} \in \mathbb{R}_+^n : a^{j\mathrm{T}} \widetilde{w} \le W,\ j = \overline{1,\eta}\}. \quad (5)$$

Note that we consider only non-negative CS because this leads to better numerical results and prevents some technical difficulties in the methods proposed below.

A linear program. We can maximize a linear objective function on $D(P(W, w))$:

$$\max\{h^{\mathrm{T}} \widetilde{w} : \widetilde{w} \in \mathbb{R}_+^n,\ a^{j\mathrm{T}} \widetilde{w} \le W,\ j = \overline{1,\eta}\} \quad (6)$$

with some $h \in \mathbb{R}_+^n$. The LP (6) can be solved by a cutting-plane algorithm, cf. [4].

The dual program to (6) is

$$\min\{W \cdot \sum_{j=1}^{\eta} x : \sum_{j=1}^{\eta} a^j x_j \ge h,\ x \ge 0\}. \quad (7)$$

For any $k \in \{1, \dots, d\}$ and corresponding data $(W, w) = (W_k, w^k)$, $h = (h_i)_{i=1}^n = (\prod_{k' \ne k} w_i^{k'})_{i=1}^n$ this is the k*-th bar LP relaxation of OPP*, cf. [3].

Thus, any CS $\widetilde{w}$ is a feasible dual solution of (7). *Dual-feasible functions* (DFFs) produce such solutions heuristically:

Definition 2. *A function* $f : [0,1] \to [0,1]$ *is called dual-feasible if for any finite set of non-negative real numbers* $(x_1, \dots, x_m) \in \mathbb{R}_+^m$ *holds:*

$$\sum_i x_i \le 1 \Rightarrow \sum_i f(x_i) \le 1.$$

Note the difference between CS (Definition 1) and DFF: the latter are just an instrument to construct the former. For a certain OPP instance, DFFs are applied independently in each dimension without considering the main goal (2).

2.2. Iterated 1D relaxations

Our heuristic should keep the objective (2) in mind. The idea is to simplify (2) down to a linear program (6) by fixing the CS in all but one dimension. Let us consider the 2D case first. Let an OPP-2 instance be defined as $(W, H, w, h) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$. For a certain $\widetilde{h} \in \mathbb{R}^n$, a CS for (H, h), we look for the "best" $\widetilde{w}$, a CS for (W, w). This problem is modeled by (6). To obtain many different $\widetilde{w}$ and $\widetilde{h}$ (and possibly high volume bounds ($\widetilde{w}^{\mathrm{T}} \widetilde{h}$), we propose the iterative heuristic described in Figure 1.

Input: 2D OPP instance (W, H, w, h).

S0. Set $w^0 = w$, $h^0 = h$, and $t = 0$.

S1. Solve the following LPs for both dimensions:

$$w^{t+1} \in \arg\max\{h^{t\mathrm{T}} \widetilde{w} : \widetilde{w} \in D(P(W, w))\}, \quad (8a)$$

$$h^{t+1} \in \arg\max\{w^{t\mathrm{T}} \widetilde{h} : \widetilde{h} \in D(P(H, h))\}. \quad (8b)$$

S2. Compute volume bounds:
if $\max\{w^{t+1\mathrm{T}} h^t, w^{t\mathrm{T}} h^{t+1}\} > WH$
then the OPP instance (W, H, w, h) is infeasible;
else set $t \leftarrow t + 1$ and goto S1.

Fig. 1. The iterative heuristic for the 2D case

In the 2D case it is possible to show some monotonicity of the bounds, which explains the choice of tested scalar products in step S2:

Proposition 2. *Consider the 2D algorithm of Figure 1. It holds for any* $t \in \mathbb{Z}_+$*:*

$$w^{t+1^{\mathrm{T}}} h^t \geq \widetilde{w}^{\mathrm{T}} h^t, \quad \forall \widetilde{w} \in D(P(W,w)),$$

and $$h^{t+1^{\mathrm{T}}} w^t \geq \widetilde{h}^{\mathrm{T}} w^t, \quad \forall \widetilde{h} \in D(P(H,h)).$$

A heuristic for three and more dimensions is given in Figure 2.

Input: OPP-d instance $I = (W_1, \dots, W_d, w^1, \dots, w^d)$.

S0. Initialize the set of CS: $\widetilde{\Omega}^k = \{w^k\}$, $k = \overline{1,d}$.
Select the starting vectors $\widetilde{w}^{k,0} \in \widetilde{\Omega}^k$, $k = \overline{1,d}$ and set $t = 1$.

S1. For each dimension $k_0 = \overline{1,d}$, initialize the objective function:

$$\widetilde{h}^{k_0,t} = (\widetilde{h}_i^{k_0,t})_{i=1}^n = \Big(\prod_{k \neq k_0} \widetilde{w}_i^{k,t-1} \Big)_{i=1}^n.$$

S2. For each dimension $k = \overline{1,d}$ do:

a) Solve

$$\widetilde{w}^{k,t} \in \arg\max \{ \widetilde{h}^{k,t\mathrm{T}} \widetilde{w} : \widetilde{w} \in D(P(W_k, w^k)) \} \quad (9)$$

b) **If** $\widetilde{w}^{k,t} \in \widetilde{\Omega}^k$ (repeated CS)
then replace $\widetilde{w}^{k,t}$ using DFF $u^{(p)}$ [10] with some $p \in \mathbb{N}$.
Add $\widetilde{w}^{k,t}$ to $\widetilde{\Omega}^k$.

S3. Compute the volume bounds from all known CS:
if $\max\{\sum_i \prod_k \widetilde{w}_i^k : \widetilde{w}^k \in \widetilde{\Omega}^k, k = \overline{1,d}\} > \prod_k W_k$
then the OPP instance I is infeasible.
Else set $t \leftarrow t+1$ and goto S1.

Fig. 2. The iterative heuristic for three and more dimensions

This scheme allows many variations. For example, we can replace steps S1--S2 by the so-called spiral scheme shown in Figure 3.

S1–S2. For each dimension $k_0 = \overline{1,d}$ do:

a) Initialize the objective function: $\widetilde{h}^{k_0,t} = (\widetilde{h}_i^{k_0,t})_{i=1}^n = \Big(\prod_{k<k_0} \widetilde{w}_i^{k,t} \prod_{k>k_0} \widetilde{w}_i^{k,t-1} \Big)_{i=1}^n$.

b) Solve (9) with $k = k_0$.

c) **If** $\widetilde{w}^{k_0,t} \in \widetilde{\Omega}^{k_0}$ (repeated CS)
then replace $\widetilde{w}^{k_0,t}$ using DFF $u^{(p)}$ [10] with some $p \in \mathbb{N}$.
Add $\widetilde{w}^{k_0,t}$ to $\widetilde{\Omega}^{k_0}$.

Fig. 3. The spiral scheme of the iterative heuristic

This variation possesses monotonicity properties like in the 2D case (Proposition 2), which leads to faster convergence. But after a few iterations the results are rather similar to the algorithm in Figure 2. A very useful option is to add more CS to Ω^k during initialization. However, we further discuss only the exact scheme of Figure 2.

Step S2 b) should prevent stalling of the heuristic. If CS in dimension k appears repeatedly, it is replaced by a new CS obtained with the DFF $u^{(p)} : [0,1] \to [0,1]$ from [10] which is defined as

$$u^{(p)}(x) = \begin{cases} x, & (p+1)x \in \mathbb{Z}, \\ \frac{1}{p} \lfloor (p+1)x \rfloor, & (p+1)x \notin \mathbb{Z} \end{cases}$$

for any $p \in \mathbb{N}$. For non-cubic instances (see Results) we used a global variable p and increased it by 1 with every usage (starting at 1). For cubic instances it proved better to have a separate counter in each dimension. This can be explained by the following observation:

$$\max\{x^3, y^3, z^3\} \geq xyz, \qquad \forall x, y, z \geq 0.$$

2.3. A local optimality criterion

Consider the following particular volume bound:

$$v_t = \sum_i \prod_k \widetilde{w}_i^{k,t}, \qquad t \in \mathbb{Z}_+. \qquad (11)$$

It has no practical interest on itself because it is dominated by the value of the LP (10, Fig. 2) for any k. However it possesses a local optimality condition:

Proposition 3. Consider the algorithm of Figure 2. If $\widetilde{w}^{k,t+1} = \widetilde{w}^{k,t}$ for some $t \in \mathbb{Z}_+$ and all $k = \overline{1,d}$, then $(\widetilde{w}^{1,t} \dots \widetilde{w}^{d,t})$ is a local maximum of (2).

3. Conclusions

We proposed an iterative heuristic to compute conservative scales for OPP. The heuristic is based on a simplification of the original multilinear optimization problem to a linear one by fixing the CS in all dimensions but one to constants. In 2D, the results are much better than from DFFs and the same as from bilinear programming while using only small time. In 3D, the results are better than from DFFs, especially for non-cubic instances.

4. Results

The experiments were performed on modern PCs. We used CLP 1.6 (*www.coin-or.org*) as a linear programming library. All running times are reported in seconds.

4.1. 2D OPP

We considered the 27 infeasible OPP instances form [7]. (In their paper, only 41 instances are cited, but the complete set has 15 feasible and 27 infeasible instances). The container size is 20×20 and the number of items $n \leq 23$. In [7] each instance was solved in a few seconds. For the 27 infeasible instances, we compared the

2D heuristic of Figure 1 with the volume bound from DFFs and with the exact algorithm of [4].

Table 1. The dual-feasible functions tested

id	the identity,
u^k $(k = 1, 2, 3, 4, 5)$	DFFs from [10],
U^ϵ $(\epsilon = 0.1, 0.2, 0.3, 0.4, 0.5)$	
ϕ^ϵ $(\epsilon = 0.1, 0.2, 0.3, 0.4)$	
f_2^k $(k = 50, 100, ..., 450)$	a DFF from [5],
f_1^k $(k = 50, 100, ..., 450)$	a *DDFF* [5].

We considered the 33 DFFs given in Table 0. The volume bound was calculated using all 33^2 pairs of DFFs as

$$v = \max_{g_j \in \{id, u^k, U^\varepsilon, \phi^\varepsilon, f_2^k, f_1^k\}} \frac{\sum_{i=1}^{n} \prod_{j=1}^{d} g_j(w_i^j)}{\prod_{j=1}^{d} g_j(W_j)}. \quad (13)$$

To compute u^k, U^ε, and ϕ^ε, the sizes were scaled so that $W_j = 1$, $j = \overline{1,d}$.

The code from [4] solved 13 instances, 12 of them rather quickly and one more (instance 00N23) after 30 seconds. Note that among the 13 instances are the 10 ones solved by the bar relaxation (iteration 0 of the heuristic). The heuristic solved the same 13 instances. The values are monotonous with a period of two iterations, in accordance with Proposition 2..

4.2. 3D OPP

For 3D tests we considered instances generated similarly to [3]. The number of items was $n = 20$ (for other values of n, the behavior is similar.) For each approximate waste percentage 0%, 2%, ..., 40% we generated 100 instances. The time to compute v(DFF) was about 1 second per instance, cf. [3].

For the non-cubic instances (side ratio 1:20), the LP[0] value mostly dominates the bound from DFFs. This dominance noticeably increases in the subsequent iterations of the heuristic. For cubic instances, the LP[0] value is very weak compared to DFFs. The weakness of LP[0] for cubes can be explained by a greater loss of geometric information. An interesting question is the relation of the sets of instances solved by DFFs and the heuristic. For non-cubic instances, all but 1 instances solved by DFFs were also solved by the heuristic. For cubic instances, 99 of them could be solved only by DFFs. For instances with maximal item side ratio 1:3, the results are similar to those with 1:20, except that only 837 instances were solved after 10 iterations.

The stalling prevention feature (step S2 (b) in Figure 2) proved very effective. Without this feature, we observed no improvement after 3-5 iterations and considerably weaker overall results. Moreover, sometimes we observed that the objective coefficient vectors $\widetilde{h}^{k,t}$ (step S1) became zero. Interestingly, this always happened for all $k = \overline{1,d}$ simultaneously. Stalling protection automatically resolved this situation.

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