Some Remarks on Simple Combinatory Calculi

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Abstract[1]

The sufficient conditions are given such that for a simple combinatory calculus C the relation

$$(\forall\vec{P})\left(Y[\vec{x}:=\vec{P}]\sim Z[\vec{x}:=\vec{P}]\right)$$

implies the syntactic equality

$$Y\equiv Z,$$

where Y, Z are any terms over the set of variables $\{x_1,...,x_m\}$, $\vec{x}=x_1,...,x_m$, $\vec{P}=P_1,...,P_m$, and P_1, ..., P_m are closed terms,. ~ is the equivalence relation of terms in the calculus C.

1. The simple combinatory calculi

It is known that the calculus *CL* (weak theory of combinators) is defined by the axioms

$$KMN=M \text{ and } SMNL=ML(NL),$$

where *K* и *S* are the combinators and *M*, *N*, *L* are arbitrary *CL*-terms. In the present paper we consider the calculi like *CL*. In the following we use the notions and the notations from [1–3].

Let *A* be a finite alphabet of *constants*, *V* be a set of *variables*, $W\subseteq A\cup V$. The set *Tm*(*W*) of *terms* over *W* is the smallest set such that:

1. $W\subseteq Tm(W)$;
2. if $P,Q\in Tm(W)$ then $(PQ)\in Tm(W)$.

In writing terms the outermost parentheses are omitted. The symbol ≡ denotes syntactic equality of terms. We write *Tm* instead of $Tm(A\cup V)$. *M*, *N*, *L*, ... is a syntactic notation for arbitrary terms in *Tm* and *x*, *y*, *z*, ... is a syntactic notation for arbitrary variable in *V*. The *length* of a term *M* is the number of symbols of $A\cup V$ in *M*. Unless otherwise stated a term is a term over the set $W=A\cup V$.

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Let $\vec{N}=N_1,...,N_k$. Then

$$MN_1...N_k\equiv M\vec{N}\equiv(...((MN_1)N_2)...),$$

(association to the left).

FV(*P*) is the set of variables in *P*, *P* is closed if *FV*(*P*)=∅. The closed terms are the terms over the set *A*. Notation $M(x_1,...,x_n)$ means that $FV(M)\subseteq\{x_1,...,x_n\}$.

M[*x*:=*N*] denotes the result of substituting *N* for the occurrences of *x* in *M*. Let $\vec{x}=x_1,...,x_k$, $\vec{N}=N_1,...,N_k$. Then

$$M[x_1,...,x_k:=N_1,...,N_k]\equiv M[\vec{x}:=\vec{N}]$$

denotes the result of simultaneous substituting N_1, ..., N_k for the occurrences of x_1, ..., x_k respectively in *M*. If $M\equiv M(x_1,...,x_k)$ then we write $M(N_1,...,N_k)$ instead of $M[\vec{x}:=\vec{N}]$.

The equality

$$ax_1...x_n=X \qquad (1)$$

where $a\in A$, X is a term over the set of variables $\{x_1,...,x_n\}$, is called a *combinatory identity*. A constant *a* is called a *basic combinator* with the identity (1). The number *n* is called the *rank* of the combinator *a* and denoted by rk(*a*).

Let Σ be a nonempty set of combinatory identities:

$$\begin{cases} a_1x_1...x_{n_1}=X_1, \\ \quad ... \\ a_rx_1...x_{n_r}=X_r, \end{cases}$$

where the basic combinators $a_i\in A$ are distinct, $n_i\geq 1$, $1\leq i\leq r$. The symbols of *A* which are different from a_1, ..., a_r are called the *atoms*.

The set Σ defines the *calculus* *C*=*C*(Σ). The formulas of the calculus *C* are the equations of terms *M* = *N* where $M,N\in Tm$. The calculus *C* has the following axioms:

$$a_1P_1...P_{n_1} = X_1(P_1,...,P_{n_1}),$$
$$...\qquad (2)$$
$$a_rP_1...P_{n_r} = X_r(P_1,...,P_{n_r}),$$

where P_1, ..., P_{n_i} are any terms, $1 \le i \le r$. The set of axioms is completed by the formulas

$$P = P,$$

where $P \in Tm$. The rules of the calculus C are as follows:

$$\frac{M = N}{N = M}, \quad \frac{M = N \quad N = L}{M = L}, \quad \frac{M = N}{ML = NL}, \quad \frac{M = N}{LM = LN}.$$

We write $C \vdash M = N$ if the formula $M = N$ is proved in the calculus C.

The calculus $C^\circ = C^\circ(\Sigma)$ is obtained from the calculus C if in formulas, axioms and rules only closed terms are used. We write $C^\circ \vdash M = N$ if the formula $M = N$ is proved in the calculus C° (here M and N are the closed terms).

The calculi C and C° are called the *simple combinatory calculi* (SCC). Such calculi are studied in [5–9]. The calculus CL (weak theory of combinators) is *combinatory complete*. An arbitrary SCC C may not be combinatory complete.

Just as in the case of the calculus CL, the axioms (2) define for the calculus C the reduction relation $\to$ on the set Tm. If $P=Q$ is one of the equations (2) then P is called a *redex* (a *redex of the calculus C*) and Q is called its *contractum* (the *value* of the redex P). We write $M \to N$ iff a term N is obtained from a term M by replacing one occurrence of a redex in M by its contractum. Transitive reflexive and transitive reflexive symmetric closure of the reduction relation $\to$ is denoted by $\to^*$ and $\sim$ respectively.

Theorem 1. $C \vdash M = N \Leftrightarrow M \sim N$.

Proof. See [1,3]. □

Theorem 2 (Church–Rosser theorem).

$$M \sim N \Leftrightarrow (\exists L)[M \to^* L \wedge N \to^* L]$$

Proof. See [1,3,4]. □

A term M is in *normal form* (NF) in the calculus C iff there is no term N such that $M \to N$.

Corollary. If the terms M and N are in NF in the calculus C then $M \sim N \Leftrightarrow M \equiv N$. □

A term M has a NF in the calculus C iff there is a term N such that:

1) N is in NF in the calculus C;

2) $M \to^* N$.

By the Church–Rosser theorem a term N (if exists) is unique for a term M. The term N is called the *normal form* of the term M.

The sequence (finite or infinite)

$$P \equiv P_0 \to P_1 \to P_2 \to ..., \qquad (3)$$

where P, P_i are terms, $i \ge 0$, is called a *reduction chain* of P. If the sequence (3) is finite and is finished by P_n, $n \ge 0$, then the number n is called the *length* of this reduction chain.

A term M is in *quasi normal form* (QNF) in the calculus C iff

$$M \equiv \xi N_1...N_k,$$

where ξ is an atom or a variable or a basic combinator of rank rk(ξ)>k ($k \ge 0$). A term M has QNF in the calculus C iff there is a term N such that:

1. N is in QNF in the calculus C;
2. $M \to^* N$.

Let a term $M \equiv aN_1...N_k$ where a is a basic combinator. The term M is *h-reduced* to a term N in the calculus C, notation $M \to_h N$, iff for some $i \le k$ the term $P \equiv aN_1...N_i$ is a redex and $N \equiv QN_{i+1}...N_k$ where a term Q is the contactum of P. We say also that the term N is obtained from the term M by one h-step. The term P is called the *head redex* of the term M. The relation $\to_h$ is called a *one step head reduction*.

The sequence (finite or infinite)

$$M \equiv M_0 \to_h M_1 \to_h M_2 \to_h \cdots$$

is called the *head reduction chain* of M. If M_n is in QNF then one say that the head reduction chain of M is finished by M_n. Otherwise one say that M has the infinite head reduction chain.

Theorem 3. Let a term M has a NF in the calculus C. Then there is a number m such that any reduction chain of M has not more than m h-steps.

Proof. See [1–3]. □

Lemma 1. Let a term $M \equiv aN_1...N_k$ has a NF in the calculus C, a be a basic combinator. Then for each i, $0 \le i \le k$, the term $M_i \equiv aN_1...N_i$ has a QNF in C.

Proof. By theorem 3 there is a number m such that any reduction chain of M has not more than m h-steps. Let us consider the head reduction chain of M_i:

$$M_i \equiv M_i^0 \to_h M_i^1 \to_h \cdots. \qquad (4)$$

Then

$$M \equiv M_i^0 N_{i+1}...N_k \to_h M_i^1 N_{i+1}...N_k \to_h \cdots. \qquad (5)$$

If the chain (4) is infinite then the chain (5) is also infinite and has more than m h-steps. We have arrived at a contradiction. Hence the head chain (4) is finite and the term M_i has a QNF in C. □

The terms $M \equiv aM_1...M_k$ $(k \geq 0)$ and $N \equiv bN_1...N_s$ $(s \geq 0)$ are in *essentially different* QNF iff

1) M and N are in QNF in the calculus C;

2) $k \neq s$ *or* $a \neq b$ *where* $a, b \in A \cup V$.

Lemma 2. If M and N are in essencially different QNF then $M \not\sim N$ in the calculus C.

Proof. By the Church--Rosser theorem. □

Let the sequence

$$Q_1, Q_2, ... \qquad (6)$$

of closed terms satisfies the following conditions:

(P1) $Q_i \not\sim Q_j$ in the calculus C for $i \neq j$;

(P2) Q_i has no QNF for each i.

Lemma 3. Let $Y \equiv Y_1 Y_2$ be a term over the set $\{x_1,...,x_m\}$, $Y_1 \equiv x_i Z_1 ... Z_k$, $Y_2 \equiv x_j U_1 ... U_s$, $k \geq 0$ $s \geq 0$, $1 \leq i, j \leq m$. If

$$Y[x_1,...,x_m := Q_1,...,Q_m] \to^* M$$

then there are the closed terms M_1 and M_2 such that

(a) $M \equiv M_1 M_2$;

(b) $Y_t[x_1,...,x_m := Q_1,...,Q_m] \to^* M_t$ (t=1,2);

(c) $M_1 \equiv SN_1...N_k$ for some closed terms $S, N_1, ..., N_k$ and $Q_i \to^* S$;

(d) $M_2 \equiv TL_1...L_s$ for some closed terms $T, L_1, ..., L_s$ and $Q_j \to^* T$.

Proof. By induction on the length of the reduction chain of $Y[x_1,...,x_m := Q_1,...,Q_m]$ to M. □

Let

$$MN^{\sim k} \equiv MN...N \qquad (N \text{ repeats } k \text{ times}).$$

Lemma 4. Let there is a closed term Q which has no QNF in the calculus C. Then there exists a sequence of closed terms (6) which satisfies the conditions (P1), (P2) and the following condition

(P3) $(\forall i)(\exists Q_i')[Q_i \equiv Q_i' a]$ where a is a basic combinator in C.

Proof. Let $Q^* \equiv QQ$. Define

$$Q_1 \equiv Q^* a, \quad Q_{m+1} \equiv Q_m a \quad (m \geq 1) \qquad (7)$$

where a is an arbitrary basic combinator of the calculus C. Then

$$Q_m \equiv QQa^{\sim m} \quad (m \geq 1).$$

The condition (P3) is satisfied. Now we prove that the sequence (7) satisfies the conditions (P1) and (P2).

Let $Q_m \to^* U$ $(m \geq 1)$. By induction on the length of the reduction chain of Q_m to U it is not difficult to prove the following three propositions:

α) $U \equiv Q' Q'' a^{\sim m}$;

β) $Q \to^* Q'$ and $Q \to^* Q''$;

γ) U is not in QNF in C.

It follows from γ) that the sequence (7) satisfies the condition (P2).

Let $Q_i \sim Q_j$ for some i and j, $i \neq j$. We may assume that $i<j$. By the Church–Rosser theorem there exists a term U such that

$$Q_i \to^* U \quad \text{and} \quad Q_j \to^* U.$$

Clealy U is a closed term. By α) and β) there are the closed terms Q_i', Q_i'', Q_j', Q_j'' such that

$$U \equiv Q_i' Q_i'' a^{\sim i} \equiv Q_j' Q_j'' a^{\sim j},$$

$$Q_i \to^* Q_i', \quad Q_i \to^* Q_i'',$$

$$Q_j \to^* Q_j', \quad Q_j \to^* Q_j''.$$

Then $Q_j'' \equiv a$ and the term Q has a QNF in the calculus C. We have arrived at a contradiction with the condition of the lemma. Hence the sequence (7) satisfies the condition (P1). □

The right side X of the combinatory identity (1) can be uniquely presented in the form

$$X \equiv x_i X_1 ... X_k$$

where X_1, ..., X_k are terms over the set $\{x_1,...,x_n\}$, $k \geq 0$. The number k is called the *degree* of a basic combinator $a \in A$ and is denoted by dg(a).

Lemma 5. Let the calculus C has a basic combinator $a \in A$ such that $\mathrm{rk}(a) \leq \mathrm{dg}(a)$. Then there exists a closed term Q which has no QNF in C.

Proof. Define $Q \equiv aa^{\sim n}$ where n=rk(a). Let $Q \to^* U$. Clearly U is a term over the set $\{a\}$, $U \equiv aU_1...U_m$ for some closed terms U_1, ..., U_m, $m \geq \mathrm{rk}(a)$. The term U

is not in QNF in C. Hence Q has no QNF in the calculus C. □

Proposition 1. Let Y and Z be any terms over the set $\{x_1,...,x_m\}$ ($m \geq 1$). Suppose that there exists a closed term Q which has no QNF in the calculus C. If for all closed terms P_1, ..., P_m,

$$Y[x_1,...,x_m := P_1,...,P_m] \sim Z[x_1,...,x_m := P_1,...,P_m]$$

then $Y \equiv Z$.

Proof. By induction on the structure of Y, using the lemmas 1–5. □

Proposition 2. Let the terms Y and Z be as in the proposition 1. Suppose that the calculus C has an atom. If for all closed terms P_1, ..., P_m,

$$Y[x_1,...,x_m := P_1,...,P_m] \sim Z[x_1,...,x_m := P_1,...,P_m]$$

then $Y \equiv Z$.

Proof. Let a be an atom of the calculus C. It is possible to construct the closed terms Q_1, ..., Q_m over the set $\{a\}$ which are distinct and have the same length. In this case the closed terms

$$Y[\vec{x} := \vec{Q}] \quad \text{and} \quad Z[\vec{x} := \vec{Q}]$$

where $\vec{x} = x_1,...,x_m$, $\vec{Q} = Q_1,...,Q_m$, are in NF in C. By the Church–Rosser theorem we have the syntactic equality

$$Y[\vec{x} := \vec{Q}] \equiv Z[\vec{x} := \vec{Q}].$$

Since Q_1, ..., Q_m have the same length, the numbers of occurrences of the variables in Y and Z are equal. By induction on the structure of Y one can prove the syntactic equality $Y \equiv Z$. □

2. Conclusion

In the present paper the sufficient conditions are given such that for a simple combinatory calculus C the relation

$$(\forall \vec{P})\left(Y[\vec{x} := \vec{P}] \sim Z[\vec{x} := \vec{P}]\right)$$

implies the syntactic equality

$$Y \equiv Z,$$

where Y, Z are any terms over the set of variables $\{x_1,...,x_m\}$, $\vec{x} = x_1,...,x_m$, $\vec{P} = P_1,...,P_m$ and P_1, ..., P_m are closed terms, ~ is the equivalence relation of terms in the calculus C.

References

1. Curry H. B., Feys R. "Combinatory logic". Vol. 1. Amsterdam, Holland, 1958.
2. Curry H. B., Hindley J. R., Seldin J. P. "Combinatory logic". Vol. 2. Amsterdam, Holland, 1972.
3. Barendregt H. P. "The Lambda Calculus. Its Syntax and Semantics". Amsterdam, Holland, 1981.
4. Church A., Rosser J. B. "Some properties of conversion". *Trans. Amer. Math. Soc.*, 1936; Vol. 39, pp. 472–482.
5. Shabunin L. V. "Simple combinatory calculi". *Vestnik Moskovskogo Universiteta. Seriya I: Matematika, Mekhanika*, 1973; № 6, pp. 30–35.
6. Shabunin L. V. "Some algorithmical problems of calculi of combinatory logic". *Vestnik Moskovskogo Universiteta. Seriya I: Matematika, Mekhanika*, 1974; № 6, pp. 36–41.
7. Shabunin L. V. "On an interpretation of α-words which is associated with simple combinatory calculi". *Vestnik Moskovskogo Universiteta. Seriya I: Matematika, Mekhanika*, 1975; № 6, pp. 3–9; 1976; № 1, pp. 12–17.
8. Shabunin L.V. "On the interpretation of combinators with weak reduction". *J. Symbol. Log.*, 1983; Vol. 48, № 3, pp. 558–563.
9. Shabunin L.V. "On computations in some combinatory calculi". In: "Proc. of the 11th Int. Workshop on Computer Science and Information Technologies (CSIT'2009)". Crete, Greece, 2009; Vol. 1, pp. 54–56.