Check of Complex Statistical Hypotheses about Values of Functions of Parameters Applying Criterion of Optimality

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Abstract[1]

The possibility of application of the variational approach for checking complex statistical hypotheses about values of functions of unknown parameters is considered here.

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Checking parametrical hypotheses is a widespread applicative statistical problem. For checking complex hypotheses the Valdovsky reduction, which based on the introduction of distribution laws of unknown parameters, is recommended. However, if weights (costs) of various errors are not equivalent, it is reasonable to apply the approach connected with creation of decision functions where it is necessary to introduce criterion of optimality of accepted decisions [1]. In paper [2] the constructive way for optimum splitting of selective area for acceptance of one of many competing parametric hypotheses is offered. It is based on the variation approach with counting various weights (costs) of errors. In practice there are tasks of checking hypotheses about values of functions of unknown parameters. For example, at audit of bank operations it can be connected not only with the analysis of probabilities of the first and the second errors, but also with necessity to account financial consequence from each accidental error. If ξ has normal distribution law and it is required to accept the hypothesis, which consists in the following: the probability $p(a)$ of inequality $\xi \le a$, where a – the given value, does not exceed the demanded value h, where h – an allowed value. In fact, here a hypothesis about value of function of parameters is being checked. This function is $f = \Phi(\frac{a-m}{\sigma})$, where m, σ^2 – mathematical expectation and dispersion of value, ξ, Φ – symbol of Laplace function. Besides, various quantitative characteristics are used in reliability theory. For example, in conditions of exponential distribution law the characteristic of reliability of the technical device can be a failure density λ and a probability $p(T) = \exp(-\lambda T)$ of non-failure operation of the device for a certain interval of time T, which is a function of parameter.

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The main problem when checking two competing hypotheses H and $\overline{H}$ about value of function $f(\theta)$ of unknown parameter (which can be a vector) takes place is to find optimum splitting of the selective area into two non-intersecting parts F and W. The splitting is so, that if selective vector ξ with components $\xi_1, \xi_2, ..., \xi_n$ matches area W, the hypothesis $\overline{H}$ is accepted. Otherwise the opposite hypothesis H is accepted. The hypothesis H consists in that, that the value of function $f(\theta)$ belongs to certain area D, and the hypothesis $\overline{H}$ – that the value of function belongs to area $\overline{D}$. Areas D and $\overline{D}$ have no shareable points and their union coincides with set of every possible values of function $f(\theta)$. In the given report the most widespread hypotheses like $f(\theta) \le h$ and the opposite $f(\theta) > h$ are considered here.

When reduction of complex hypothesis is applied the density of distribution of unknown parameter is used. In this conditions the distribution law of function of parameter can be defined. Considering a value of this function as a new parameter the task turns into checking a complex parametrical hypothesis about a value of a new parameter which can be solved with account of error's weights (costs) [2].

However it is possible to offer more simple approach which excludes finding a distribution law of function of parameter.

As to areas D and $\overline{D}$ of values of functions the areas Δ and $\overline{\Delta}$ of values of unknown parameters match, the statistical test quality can be described by average value of weights (costs) of errors which can be defined by functional $J[\varphi]$, which depends on splitting border of selective space $x_n = \varphi(x_1, x_2, ..., x_{n-1})$. This functional can be formed basing on various reasons which reflect concrete practical requirement. For example, $J[\varphi] = \alpha_1 \overline{S}_1 + \alpha_2 \overline{S}_2$, where α_1, α_2 – probabilities of the first and the second sort errors (depend on function φ), $\overline{S}_1, \overline{S}_2$ – average weight s (costs)of errors from acceptance of hypotheses $H, \overline{H}$ respectively. The functional $J(\varphi)$ can be presented in the following way:

$$J(\varphi) = [\int dx_1 ... \int dx_{n-1} \int_{\varphi}^{\infty} \overline{p}_H(x_1, ..., x_n) dx_n] \overline{S}_1 +$$

$$[\int dx_1 ... \int dx_{n-1} \int_{-\infty}^{\varphi} \overline{p}_{\overline{H}}(x_1, ..., x_n) dx_n] \overline{S}_2,$$

where

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$\overline{p}_H(x_1, x_2, ..., x_n)$, $\overline{p}_{\overline{H}}(x_1, x_2, ..., x_n)$ – density of a selective vector distribution averaged on parameters in accordance with Valdovsky reductions in hypotheses conditions $H, \overline{H}$ respectively, $p_\Delta(\theta), p_{\overline{\Delta}}(\theta)$ – density of parameter distribution in hypotheses conditions $H, \overline{H}$. The optimum splitting border of selective space is found by getting φ from the following equation [2]:

$\overline{S}_1 \overline{p}_H(x_1, x_2, ..., x_{n-1}, \varphi) = \overline{S}_2 \overline{p}_{\overline{H}}(x_1, x_2, ..., x_{n-1}, \varphi)$.Values $\overline{S}_1, \overline{S}_2$ depend on accepted expressions for describing weights (costs) of errors. Accepting that $\overline{S}_1$ is defined by possible values of parameter within area Δ, and $\overline{S}_2$ – values of parameter within area $\overline{\Delta}$, $\overline{S}_1, \overline{S}_2$ change that lead to other splitting of selective space.

Let's consider the algorithm of acceptance of complex hypotheses for function $f = \exp(-\lambda T)$ of parameter λ of exponential distribution law when h is given. Hypothesis H : the inequality $a \le \lambda \le h$ takes place, and hypothesis $\overline{H}$: the inequality $h < \lambda \le b$ takes place. In this case for finding function $x_n = \varphi(x_1, x_2, ..., x_{n-1})$ the following equation is used:

$$\overline{S}_1 \int_a^h p_\Delta(\lambda) \lambda^n \exp\{-S\lambda\} d\lambda =$$

$$\overline{S}_2 \int_h^b p_{\overline{\Delta}}(\lambda) \lambda^n \exp\{-G\lambda\} d\lambda,$$

where $G = \sum_{i=1}^{n} x_i$.

Finding G from this inequality, we will get obvious expression for function:

$$x_n = \varphi(x_1, x_2, ..., x_{n-1}) = G - x_1 - x_2 - ... - x_{n-1}.$$

Variable G has the sense of boundary value of the sum of selective values, which excess leads to acceptance of the hypothesis $\overline{H}$. Integrals in expression (1) can be calculated analytically, if distribution densities $p_\Delta(\lambda), p_{\overline{\Delta}}(\lambda)$ are described by polynomial. The equation (1) is non-linear towards G and have to be solved by means of numerical procedures. For this the special program was prepared which application is illustrated in work [3].

It is necessary to mark, that hypotheses H_1: $\lambda \le h$ and $H_2 : \exp\{-\lambda h\} \le \exp\{-\lambda T\}$ are not equivalent at the identical analytical description of weight (costs) of value errors of function of parameter and of weight (costs) of error at defining parameter. Therefore the splittings of selective area for these cases will differ. The similar situation takes place at prediction of values of function $\eta = f(\xi)$ of a random variable ξ if prediction is considered as acceptance of decision, which consists in a choice of such a value $\hat{\eta} = q$ that minimizes average weight of error.

Let's consider the following example. Symbol $p_\xi(x)$ describes the distribution density of a random variable ξ. It is required to make a decision (to predict) about value of function of this random variable ξ. Let's choose an average deviation square $J = M[(q - \eta)^2]$ as a criterion of optimality. Certainly, there are no reasons to hope that the estimation $q = \hat{\eta} = f(\hat{\xi})$, where $\hat{\xi}$ is built up using the minimum average deviation square criterion, will be optimum. As, when $\hat{\eta} = q$, the following equality takes place:

$$J = \int [q - f(x)]^2 p_\xi(x) dx =$$

$$q^2 - 2qM[f(\xi)] + M[f^2(x)],$$

the optimum value , got from the condition , is . However in general the value doesn't coincide with the value , if . Particularly, if , , where deviation dispersion of variable and – optimum estimation.

4. Conclusion

For checking complex statistical hypotheses about values of functions of unknown parameters variational approach can be applied.

References

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