# Irregular polyomino tiling using integer programming

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## Abstract[1]

This paper is devoted to the problem of polyomino tiling. Two cases discussed: tiling with L-shaped trominoes and tiling with L-shaped tetrominoes. The problem is considered as an integer programming model. The mathematical model is described. This problem can be applied to the phased array design where polyomino-shaped subarrays are used to reduce the cost of the array and to avoid the regularity of antenna structure. Simulations results are presented in order to evaluate performance of the approach.

*Keywords*: optimization problems, phased array design, integer programming, polyomino.

## 1. Introduction

According to Wäscher et al. the "placement problem" is one of the cutting and packing problems category in which a weakly heterogeneous assortment of small items has to be assigned to a given, limited set of large objects [1]. The total size of accommodated small objects has to be maximised. In our work we consider the two-dimensional polyomino placement problem. Golomb explains the term polyominoes as shapes made by connecting certain numbers of equal-sized squares, each joined together with at least one other square along an edge (Fig. 1) [2]. Since 1953 when he introduced this term the problem of tiling with polyominoes has been investigated by many researchers. Moore and Robson argue that the problem of tiling finite region with unrestricted number of copies of each polyomino is NP-complete [3].

Polyomino tiling has various fields of application from mass-production to military industries. One of them refers to phased array design.

Phased array antennas consist of multiple stationary antenna elements, which are fed coherently and use variable phase or time-delay control at each element to scan a beam to given angles in space [4]. This phase controls and time delay devices are the most important parts of phased array, which make it possible to control the beam direction and to keep array pattern stationary. But for reasons of economy it's better to reduce the number of these devices in antenna array. To resolve this problem subarray technology can be used. It means that array elements are grouped into subarrays and controls are introduced at the subarray level. But using rectangular subarrays causes periodicity and radiate discrete sidelobes called quantization sidelobes. Sidelobe is a beam that represents undesirable radiation in a direction different from the main direction of the antenna. Mailloux et al. argues that using irregular polyomino subarrays can result in a major decrease in sidelobes while presenting only a few tenths of a dB gain reduction compared to rectangular subarrays [5].

So there is a problem of tiling the antenna aperture with irregular polyomino-shaped subarrays. In our work we consider tiling with L-tromino and L-tetromino figures using integer linear programming methods.

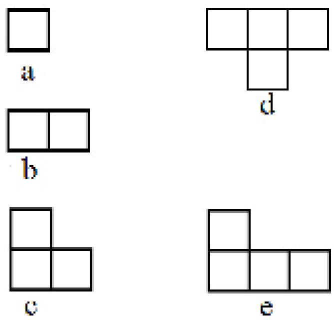


**Fig. 1. Polyominoes: monomino (a), domino (b), L-tromino (c), T- tetromino (d), L-tetromino (e).**

### State of the art

One of the many approaches in polyomino tiling problem is using heuristic methods like the genetic algorithm. It was shown in Gwee and Lim studies [6]. Their algorithm was tested in the phased array design by Chirikov et al. [7]. Chirikov also have presented their own approach called Snowball algorithm which showed better results. In the case of tiling with L-tromino using the Gwee and Lim approach the peak sidelobe level (SLL) is –27.11 dB for r = 1.3. While the Snowball algorithm presented SLL = –29.1 dB for r = 1.3. We do not compare other results

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because we consider only L-tromino and L-tetromino shapes for tiling.

Another study that shows the problem of polyomino packing problem is the work of Joachim Wollfram where he presents an exact tree search procedure for the case of packing copies of one polyomino into a rectangle and describes tabu search heuristics for packing of n distinct polyominoes [8].

Another kind of approaches refers to using mathematical programming. Galiev and Karpova considered the minimal covering problem as a linear integer program [9]. Karademir used integer programming in phased array design application. He formulated the irregular polyomino tiling problem as a nonlinear exact set covering model, where irregularity of a tiling is incorporated into the objective function using the information-theoretic entropy concept [10]. The phased array antenna simulations presented in the survey include only the case of tiling with octomino shapes.

## 2. Problem statement

We consider the tiling of the finite, rectangular region with given polyominoes, without any restriction on their number. Each polyomino can be rotated by 90 degrees and mirror-flipped. For example, there are four orientations of L-tromino and eight orientations of L-tetromino.

So, there is an n×n element region and infinite number of polyominoes. The problem is to find optimized layout of polyomino considering two following requirements:

- Minimize the number of empty spaces;
- Maximize irregularity, i.e. eliminate periodicity of polyominoes layout.

Irregularity of a structure tiled with polyominoes refers to the absence of patterns repeated with some spatial periodicity.

## 3. Mathematical formulation for tiling with L-tetromino

Considered problem can be formulated using an integer programming model. Let an n×n element structure be represented as the set of binary variables. There are eight orientations of L-tetromino (Fig.2.) with eight corresponding binary variables:

$$z_{ij} \in \{0,1\}, d_{ij} \in \{0,1\}, s_{ij} \in \{0,1\}, w_{ij} \in \{0,1\},$$

$$r_{ij} \in \{0,1\}, g_{ij} \in \{0,1\}, q_{ij} \in \{0,1\}, v_{ij} \in \{0,1\},$$

where $i = 1,\dots,n$ and $j = 1,\dots,n$ are coordinates of polyominoes on the structure.

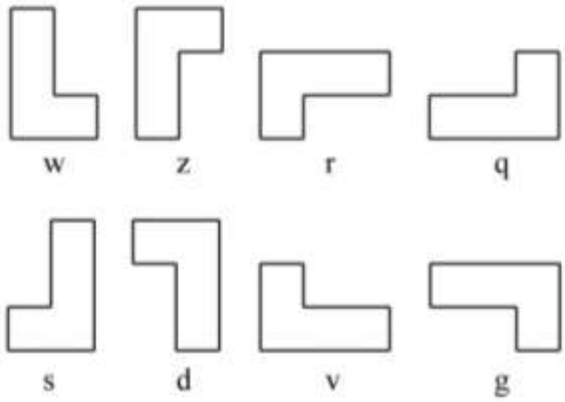


**Fig 2. Eight orientations of L-shaped tetromino.**

The objective is to maximize the sum of eight variables:

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n}(z_{ij} + d_{ij} + s_{ij} + w_{ij} + r_{ij} + q_{ij} + v_{ij} + g_{ij})$$

subject to:

$$z_{ij}+z_{i+1,j}+z_{i,j+1}+z_{i+2,,j} \le 1, \quad i=1,\dots,n\text{-}2;\ j=1,\dots,n\text{-}1$$

$$d_{ij}+d_{i+1,j+1}+d_{i,j+1}+d_{i,+2,j+1} \le 1, \quad i=1,\dots,n\text{-}2;\ j=1,\dots,n\text{-}1$$

$$s_{i+1,j+1}+s_{i+1,j}+s_{i,j+1}+s_{i,-1,j+1} \le 1, \quad i=2,\dots,n\text{-}1;\ j=1,\dots,n\text{-}1$$

$$w_{ij}+w_{i+1,j}+w_{i+1,j+1}+w_{i-1,j} \le 1, \quad i=2,\dots,n\text{-}1;\ j=1,\dots,n\text{-}1$$

$$r_{ij}+r_{i+1,j}+r_{i,j+1}+r_{i,,j+2} \le 1, \quad i=1,\dots,n\text{-}1;\ j=1,\dots,n\text{-}2$$

$$g_{ij}+g_{i+1,j+1}+g_{i,j+1}+g_{i,,j-1} \le 1, \quad i=1,\dots,n\text{-}1;\ j=2,\dots,n\text{-}1$$

$$q_{i+1,j+1}+q_{i+1,j}+q_{i,j+1}+q_{i,+1,j-1} \le 1, \quad i=1,\dots,n\text{-}1;\ j=2,\dots,n\text{-}1$$

$$v_{ij}+v_{i+1,j}+v_{i+1,j+1}+v_{i+1,j+2} \le 1, \quad i=1,\dots,n\text{-}1;\ j=1,\dots,n\text{-}2$$

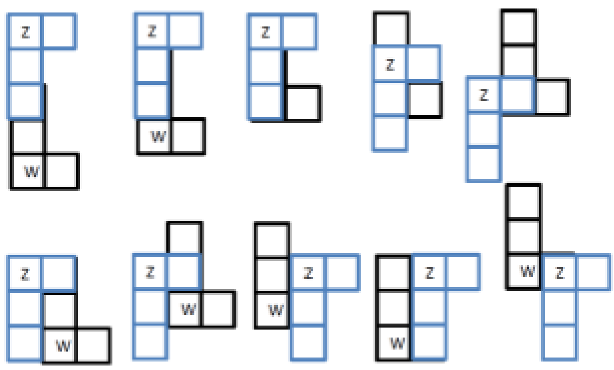

**Fig 3. Different combinations of two L-tetromino orientations.**

In order to eliminate some undesirable crossings and overlapping it was included a set of additional constraints. For example, following constraint is used to eliminate overlapping of polyominoes in one point of the structure:

$$z_{ij}+d_{ij}+s_{ij}+w_{ij}+r_{ij}+q_{ij}+v_{ij}+g_{ij}\le 1,\ (i,j)=1,\ldots,n$$

Following constraints are the example of excluding unacceptable intersections of figure d in relation to figure z (Fig. 3.):

$$z_{ij}+w_{i+2,j}\le 1,\ \ i=1,\ldots,n-2,\ j=1,\ldots,n$$

$$z_{ij}+w_{i+3,j}\le 1,\ \ i=1,\ldots,n-3,\ j=1,\ldots,n$$

$$z_{ij}+w_{i+4,j}\le 1,\ \ i=1,\ldots,n-4,\ j=1,\ldots,n$$

$$z_{ij}+w_{i+1,j+1}\le 1,\ \ i=1,\ldots,n\text{-}1,\ j=1,\ldots,n\text{-}1$$

$$z_{ij}+w_{i+2,j+1}\le 1,\ \ i=1,\ldots,n\text{-}2,\ j=1,\ldots,n\text{-}1$$

$$w_{ij}+z_{i-1,j+1}\le 1,\ \ i=2,\ldots,n,\ j=1,\ldots,n\text{-}1$$

$$w_{ij}+z_{i-2,j+1}\le 1,\ \ i=3,\ldots,n,\ j=1,\ldots,n\text{-}1$$

$$z_{ij}+w_{i,j+1}\le 1,\ \ i=1,\ldots,n,\ j=1,\ldots,n\text{-}1$$

$$z_{ij}+w_{i+1,j}\le 1,\ \ i=1,\ldots,n\text{-}1,\ j=1,\ldots,n$$

$$w_{ij}+z_{i,j+1}\le 1,\ \ i=1,\ldots,n,\ j=1,\ldots,n\text{-}1$$

There is a problem of fitting polyominoes into the boundaries of a structure (Fig.4.). Following constraints solve this problem:

$$z_{in}=0,\ \ i=1,\ldots,n,$$

$$z_{nj}=0,\ \ j=1,\ldots,n,$$

$$z_{n-1,j}=0,\ \ j=1,\ldots,n,$$

$$d_{nj}=0,\ \ j=1,\ldots,n,$$

$$d_{i1}=0,\ \ i=1,\ldots,n,$$

$$d_{n-1,j}=0,\ \ j=1,\ldots,n,$$

$$w_{in}=0,\ \ i=1,\ldots,n,$$

$$w_{1j}=0,\ \ j=1,\ldots,n,$$

$$w_{2j}=0,\ \ j=1,\ldots,n,$$

$$s_{i1}=0,\ \ i=1,\ldots,n,$$

$$s_{1j}=0,\ \ j=1,\ldots,n,$$

$$s_{2j}=0,\ \ j=1,\ldots,n,$$

$$r_{in}=0,\ \ i=1,\ldots,n,$$

$$r_{nj}=0,\ \ j=1,\ldots,n,$$

$$r_{i,n-1}=0,\ \ i=1,\ldots,n,$$

$$v_{in}=0,\ \ i=1,\ldots,n,$$

$$v_{1j}=0,\ \ j=1,\ldots,n,$$

$$v_{i,n-1}=0,\ \ i=1,\ldots,n,$$

$$g_{nj}=0,\ \ j=1,\ldots,n,$$

$$g_{i2}=0,\ \ i=1,\ldots,n,$$

$$g_{i1}=0,\ \ i=1,\ldots,n,$$

$$q_{i1}=0,\ \ i=1,\ldots,n,$$

$$q_{1j}=0,\ \ j=1,\ldots,n,$$

$$q_{i2}=0,\ \ i=1,\ldots,n.$$

The whole mathematical model for the case of tiling with L-tetromino includes 316 constraints.

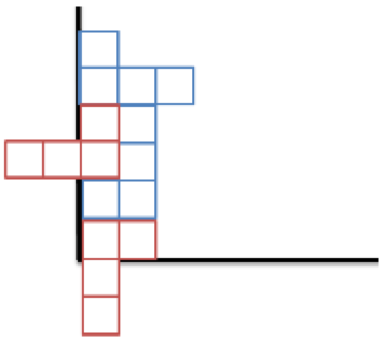

**Fig 4. Fitting the structure boundaries.**

## 4. Implementation results

Presented mathematical model for tiling with L-trominoes and L-tetrominoes has been implemented in program using Python and IBM ILOG CPLEX solver. Obtained irregular polyomino tilings were used to simulate antenna performance. Here we describe simulation results for two cases.

### 4.1. Subproblem 1. L-tromino

In the first case we tile L-trominoes on the 32×32 size structure. We don’t divide the structure on smaller sections because the computing time for the whole structure is less than one minute. Main figures including phased array parameters simulation are shown in Table 1. The program was tested on 10-20 iterations for each case of random figures placed on structure (10-30 figures) in order to find the lowest peak sidelobe level. For the better comparison only structures with one hole were selected (fullness of the structure = 99.9%).

**Table 1. Resulting figures for the first subproblem.**

| Number of random figures | 12 | 16 | 20 | 24 | 28 |
|---|---|---|---|---|---|
| Peak sidelobe level, r=1.3, dB | -28.53 | -28.64 | -28.59 | -29.15 | -28.24 |
| Peak sidelobe level, r=1.82, dB | -21.38 | -21.83 | -21.30 | -21.96 | -21.13 |
| Time, sec | 12.86 | 18.70 | 4.81 | 38.89 | 6.31 |
| Number of experiments | 30 | 15 | 10 | 10 | 10 |

It is clear that there is no definite relation between number of random figures preset and antenna performance. The tiled structure of the best peak sidelobe level (–29.15 dB for r = 1.3) is shown in Fig. 5.

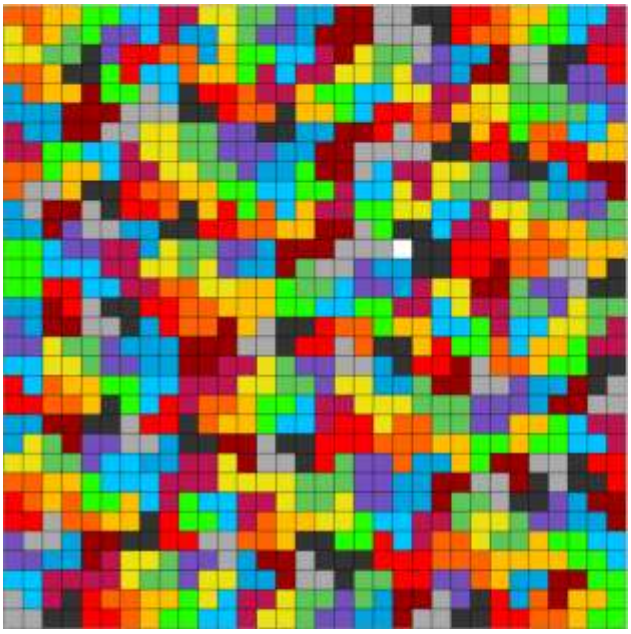

**Fig 5. 32×32 element array tiled with L-trominoes.**

The best result showed peak sidelobe level of –29.05 dB for r = 1.3, which is close to the Snowball algorithm and much better than Gwee-Lim algorithm results presented by Chirikov et al. research [7] (see Table 2).

**Table 2. Comparison of three algorithms. N=32, tiling with L-tromino.**

| Parameters | Integer Program-ming | Snowball algorithm | Gwee-Lim algorithm |
|---|---|---|---|
| Peak sidelobe level, *r* = 1.3, dB | -29.15 | -29.18 | -27.65 |
| Peak sidelobe level, *r* =1.82, dB | -21.96 | -22.36 | -20.96 |
| Fullness of the structure, % | 99.99 | 99.71 | 94.63 |

## 4.2. Subproblem 2. L-tetromino

In the second case we tile the 32×32 size structure with L-tetrominoes. The computing time for the whole structure becomes more than 10 minutes then we divide the whole structure on smaller rectangular segments of 16×16 elements size. The segmentation order is shown on Fig.6.

Antenna performance simulation results are presented for the whole 32×32 structure (see Table 3). The program was tested on 10 iterations for each case differing by the number of random figures placed on structure (16-24 figures). For the better comparison only structures with 100% fullness were selected. This case also proves that there is no relation of irregularity with number of randomly preset figures. The tiled structure of the best peak sidelobe level (–25.69 dB for r = 1.3) is shown in Fig. 7.

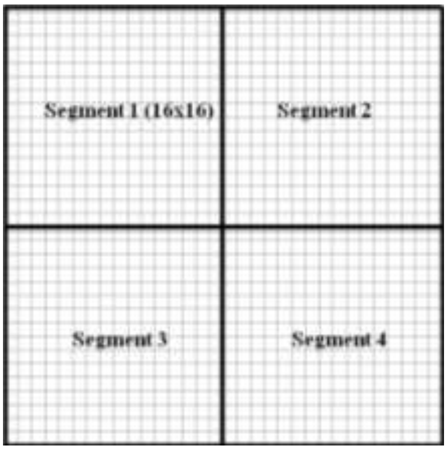


**Fig 6. Segmentation order.**

**Table 3. Resulting figures for the second subproblem.**

| Number of random figures | | 16 | 20 | 24 |
|---|---|---|---|---|
| Peak sidelobe level, *r* = 1.3, dB | | -25.63 | -25.69 | -25.33 |
| Peak sidelobe level, *r* =1.82, dB | | -16.45 | -17.41 | -16.88 |
| Time, sec | Segment 1 | 3.28 | 8.97 | 3.05 |
| | Segment 2 | 3.98 | 4.86 | 4.24 |
| | Segment 3 | 4.56 | 5.31 | 6.03 |
| | Segment 4 | 3.88 | 3.48 | 4.61 |

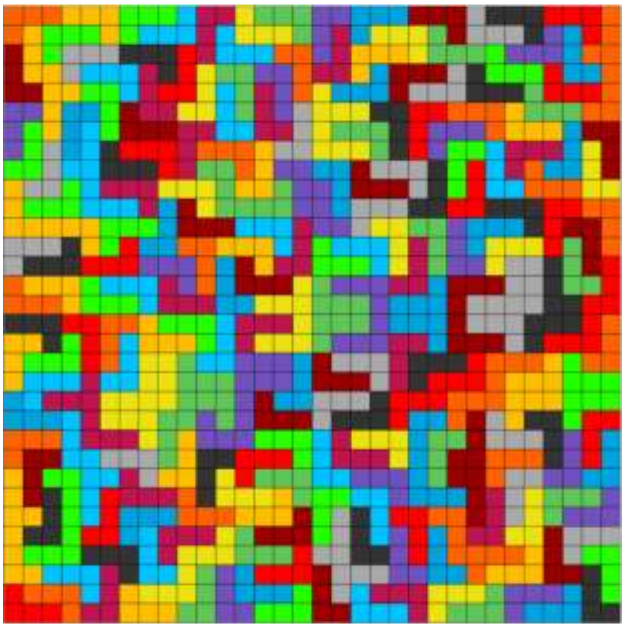

**Fig 7. 32×32 element array tiled with L-tetrominoes.**

## 5. Conclusion

The problem of tiling with polyomino is considered using an integer programming methods. The mathematical model for tiling with L-tetromino is described. Presented model was tested in the field of antenna design.

Simulation results show that the approach is relevant to the phased array design and it is possible that further iterations will perform better reduction of sidelobe level.

Presented approach for polyomino tiling can be implemented to other fields of application. Also it can be tested with more complex polyomino shapes like L-octomino and using more than one type of polyomino on the structure.

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